DESCRIPTION OF DECOHERENCE BY MEANS OF TRANSLATION-COVARIANT MASTER EQUATIONS AND LÉVY PROCESSES

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Translation-covariant Markovian master equations used in the description of decoherence and dissipation are considered in the general framework of Holevo's results on the characterization of generators of covariant quantum dynamical semigroups. A general connection between the characteristic function of classical Lévy processes and loss of coherence of the statistical operator describing the center of mass degrees of freedom of a quantum system interacting through momentum transfer events with an environment is established. The relationship with both microphysical models and experimental realizations is considered, focusing in particular on recent interferometric experiments exploring the boundaries between classical and quantum world.

Keywords: Lévy processes; decoherence; quantum dynamical semigroups

1. Introduction

A natural standpoint about quantum mechanics, which is however not the one usually considered in textbooks written for physics students, is to look at it as a new probability theory, different and reacher than the classical one[1]. This point of view becomes mandatory or at least very fruitful if one is faced with more advanced research topics, such as the description of open quantum systems or quantum information and communication theory (for a general reference see[2,3]). In these fields tools and concepts obtained relying on a probabilistic approach, also working in direct analogy with classical probability theory, have become of paramount importance. An example in this direction is given by quantum dynamical semigroups, which provide the quantum generalization of classical Markov semigroups. The subject has been the object of active research in the mathematics, physics and chemistry community over decades by now, but it is still of great interest. In particular covariance properties of such mappings under translations have been considered in detail only recently. Besides a mathematical characterization[4–7] also the actual physical relevance[8–10] of such covariant quantum dynamical semigroups has been considered.

In the present contribution we will focus mainly on the application of such translation-covariant quantum dynamical semigroups to the study and the description of the phenomenon called decoherence in the physics literature[11,12]. By such a term a whole variety of situations is meant, all having in common a loss of typical quantum interference capability, arising as a dynamical consequence of interaction of the system of interest with some other, typically much bigger, system. The phenomenology of decoherence is ubiquitous when considering open quantum systems, but its actual quantitative study requires very special experimental conditions, which can be realized e.g. in interferometric setups for massive particles, observing loss of interference fringes as a consequence of external disturbance, arising because the approximation of isolation of the system is no more realistic. For a quantitative study of the phenomenon it is in fact crucial that such decoherence effects can actually be engineered, so that their strength is under the control of the experimenter.

The paper is organized as follows. In Sect. 2 we briefly sketch the formal expression of the generator of a translation-covariant quantum dynamical semigroup. In Sect. 3 we show how such a general structure in a suitable limit can account for decoherence behaviors quantitatively described by means of the characteristic function of a classical Lévy process. In Sect. 4 we further explore how a particular physical example of realization of such generators applies to the description of decoherence in both position and momentum space, finally mentioning possible extension of the formalism in Sect. 5.

2. Translation-covariant master equations

Provided memory effects can be neglected, quantum dynamical semigroups[13,14] give a general setting for the description of the dynamics of an open quantum system[15]. In the physical literature major efforts have been devoted to the derivation or phenomenological assessment of possible generators of such quantum dynamical semigroups, so called master equations. The typical benchmark is the Lindblad structure of such generators, which goes back to the work of Gorini, Kossakowski and Sudarshan[16] and of Lindblad[17], holding true for a generator given by a bounded mapping. Attention was later devoted to possible constraints on the structure of such generators arising as a consequence of symmetry (see e.g.[18] for references).

In this respect the results of Holevo for symmetry under translations are of particular importance because of the many possible physical applications, especially in connection with typical quantum phenomena such as decoherence.

We first consider the general expression of formal generators of translation-covariant quantum dynamical semigroups as obtained by Holevo[4–6,19]. The covariance of the mapping corresponds to the requirement that its action has to commute with the unitary representation of translations on the Hilbert space of interest. The physical system we are going to consider is the centre of mass of a particle in free space, so that $\mathcal{H} = L^2\left(\mathbb{R}^3\right)$. Let $\mathcal{L}'$ be the mapping describing the dynamics in Heisenberg picture, thus acting on an observable A. In order to be covariant $\mathcal{L}'$ has to satisfy the requirement

$$\mathcal{L}'\left[\mathrm{e}^{i\boldsymbol{A}\cdot\mathsf{P}/\hbar}\mathsf{A}\mathrm{e}^{-i\boldsymbol{A}\cdot\mathsf{P}/\hbar}\right] = \mathrm{e}^{i\boldsymbol{A}\cdot\mathsf{P}/\hbar}\mathcal{L}'\left[\mathsf{A}\right]\mathrm{e}^{-i\boldsymbol{A}\cdot\mathsf{P}/\hbar} \qquad \forall\boldsymbol{A}\in\mathbb{R}^3, \tag{1}$$

where P denotes the momentum operator of the massive particle. The general structure of generator complying with this requirement is given by the formal operator expression

$$\mathcal{L}'\left[\mathsf{A}\right] = \frac{i}{\hbar}[\mathsf{H}\left(\mathsf{P}\right), \mathsf{A}] + \mathcal{L}_G[\mathsf{A}] + \mathcal{L}_P[\mathsf{A}], \tag{2}$$

where the symbols G and P denote a Gaussian and a Poisson component, the names arising from the connection with the classical Lévy-Khintchine formula. One has in particular for the Gaussian component

$$\begin{aligned}\mathcal{L}_G[\mathsf{A}] = \frac{i}{\hbar}\Bigg[&\mathsf{Y}_0 + \frac{1}{2i}\sum_{k=1}^{3}\Big(\mathsf{Y}_k L_k\left(\mathsf{P}\right) - L_k^\dagger\left(\mathsf{P}\right)\mathsf{Y}_k\Big), \mathsf{A}\Bigg]\\ &+\frac{1}{\hbar}\sum_{k=1}^{3}\Big[\left(\mathsf{Y}_k + L_k\left(\mathsf{P}\right)\right)^\dagger\mathsf{A}\left(\mathsf{Y}_k + L_k\left(\mathsf{P}\right)\right)\\ &-\frac{1}{2}\Big\{\left(\mathsf{Y}_k + L_k\left(\mathsf{P}\right)\right)^\dagger\left(\mathsf{Y}_k + L_k\left(\mathsf{P}\right)\right), \mathsf{A}\Big\}\Big]\end{aligned}$$

where $\mathsf{Y}_j = \sum_{i=1}^{3} a_{ji}\mathsf{X}_i$ with $a_{ji}\in\mathbb{R}$ for $j = 0, 1, 2, 3$, that is to say it is a linear combination of the three position operators of the test particle, appearing at most quadratically, while for the Poisson component

$$
\begin{aligned}
\mathcal{L}_P[\mathsf{A}] = & \int \mathrm{d}\mu\left(\boldsymbol{Q}\right) \sum_j \Big[L_j^\dagger\left(\mathsf{P};\boldsymbol{Q}\right) \mathrm{e}^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \mathsf{A} \mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} L_j\left(\mathsf{P};\boldsymbol{Q}\right) \\
& -\frac{1}{2}\left\{ L_j^\dagger\left(\mathsf{P};\boldsymbol{Q}\right) L_j\left(\mathsf{P};\boldsymbol{Q}\right), \mathsf{A} \right\} \Big] \\
& + \int \mathrm{d}\mu\left(\boldsymbol{Q}\right) \sum_j \Big[\omega_j\left(\boldsymbol{Q}\right) L_j^\dagger\left(\mathsf{P};\boldsymbol{Q}\right) \left(\mathrm{e}^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \mathsf{A} \mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} - \mathsf{A} \right) \\
& + \left(\mathrm{e}^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \mathsf{A} \mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} - \mathsf{A} \right) L_j\left(\mathsf{P};\boldsymbol{Q}\right) \omega_j^\star\left(\boldsymbol{Q}\right) \Big] \\
& + \int \mathrm{d}\mu\left(\boldsymbol{Q}\right) \left|\omega_j\left(\boldsymbol{Q}\right)\right|^2 \sum_j \left[\mathrm{e}^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \mathsf{A} \mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} - \mathsf{A} - \frac{i}{\hbar} \frac{[\mathsf{A}, \boldsymbol{Q}\cdot\mathsf{X}]}{1+Q^2/Q_0^2} \right].
\end{aligned}
$$

Such expressions can cover a huge variety of physical situations, accounting for both dissipative and decoherence effects. Some rough insight can be gained considering the dummy integration label $\boldsymbol{Q}$ as a momentum. The dynamics of the open system, in our case the centre of mass of a tracer particle, is thus described by an interaction only characterized by the momentum transfers between system and environment, taking place e.g. as a consequence of collisions, thus complying with translational invariance. The unitary operators $\exp\left(i\boldsymbol{Q}\cdot\mathsf{X}/\hbar\right)$ appearing in the Poisson part describe in fact a momentum kick, with rates which are not only given by functions of the momentum transfer $\boldsymbol{Q}$ itself, but also depend on the momentum operator P, thus becoming dynamic quantities. This is in particular necessary in order to correctly describe phenomena like energy transfer and approach to equilibrium. The Gaussian part corresponds to a dynamics arising as a consequence of a big number of small momentum transfers, leading to a diffusive behavior.

An interesting limiting situation appears if we neglect dissipative effects and therefore the dynamics of the momentum operator, apart from its appearance in the free kinetic term, so that, also switching to the preadjoint mapping in Schrödinger picture, and assuming $\mu\left(\boldsymbol{Q}\right)$ to be absolutely continuous with respect to the Lebesgue measure, the two contributions can be written

$$\mathcal{L}_G[\rho] = -\frac{i}{\hbar}\sum_{i=1}^{3} \mathbf{b}_i[\mathsf{X}_i, \rho] - \sum_{i,j=1}^{3} \frac{1}{2} \boldsymbol{D}_{\mathrm{ij}}[\mathsf{X}_i, [\mathsf{X}_j, \rho]] \tag{3}$$

$$\mathcal{L}_P[\rho] = \int d\boldsymbol{Q} |\lambda(\boldsymbol{Q})|^2 \left[e^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \rho e^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} - \rho - \frac{i}{\hbar}\frac{[\boldsymbol{Q}\cdot\mathsf{X}, \rho]}{1+Q^2/Q_0^2} \right] \tag{4}$$

where $\mathbf{b} \in \mathbb{R}, \boldsymbol{D} \geq 0,$ and the integration measure satisfies the Lévy condition

$$\int d\boldsymbol{Q}|\lambda(\boldsymbol{Q})|^2 \frac{Q^2}{1+Q^2} < \infty. \tag{5}$$

It is very convenient to write the contributions given by Eq. (3) and Eq. (4) in the position representation, leading to the simple expression

$$\langle \boldsymbol{X}|\mathcal{L}_G[\rho] + \mathcal{L}_P[\rho]|\boldsymbol{Y}\rangle = -\Psi\left(\boldsymbol{X}-\boldsymbol{Y}\right)\langle \boldsymbol{X}|\rho|\boldsymbol{Y}\rangle, \tag{6}$$

where according to Eq. (3) and Eq. (4) we have introduced the function

$$\begin{aligned}\Psi\left(\boldsymbol{X}-\boldsymbol{Y}\right) = {} & \frac{i}{\hbar}\mathbf{b}\cdot(\boldsymbol{X}-\boldsymbol{Y}) + \frac{1}{2}\left(\boldsymbol{X}-\boldsymbol{Y}\right)^T\cdot\boldsymbol{D}\cdot(\boldsymbol{X}-\boldsymbol{Y}) \\ & - \int d\boldsymbol{Q}|\lambda(\boldsymbol{Q})|^2\left[e^{i\boldsymbol{Q}\cdot(\boldsymbol{X}-\boldsymbol{Y})/\hbar} - 1 - \frac{i}{\hbar}\frac{\boldsymbol{Q}\cdot(\boldsymbol{X}-\boldsymbol{Y})}{1+Q^2/Q_0^2}\right],\end{aligned} \tag{7}$$

only depending on the difference $\boldsymbol{X}-\boldsymbol{Y}$ due to translational invariance. The action of the contributions given by Eq. (3) and Eq. (4) in the position representation is therefore very simple, it only amounts to multiplying the matrix elements of the statistical operator by a function of the particular form (7), whose general properties as we shall see naturally account for a description of decoherence.

3. Decoherence and Lévy processes

The master equation corresponding to Eq. (3) and Eq. (4) can be easily solved in the position representation, giving a dynamics which only changes the initial statistical operator by a multiplicative time dependent factor

$$\langle \boldsymbol{X}|\rho_t|\boldsymbol{Y}\rangle = \mathrm{e}^{-t\Psi(\boldsymbol{X}-\boldsymbol{Y})}\langle \boldsymbol{X}|\rho_0|\boldsymbol{Y}\rangle. \tag{8}$$

A key point is now the observation that Eq. (7) actually gives the general expression of the characteristic exponent appearing in the characteristic function of a Lévy process, corresponding to the celebrated Lévy-Khintchine formula[20]. As a consequence the function

$$\Phi\left(t, \boldsymbol{X}-\boldsymbol{Y}\right) = \mathrm{e}^{-t\Psi(\boldsymbol{X}-\boldsymbol{Y})} \tag{9}$$

gives the general possible expressions for the characteristic function of a classical Lévy process, different processes, e.g. Gaussian, Poisson, compound Poisson or Lévy stable processes arising corresponding to the different possible values of $\mathbf{b}, \boldsymbol{D}$ and of the positive weight $|\lambda(\boldsymbol{Q})|^2$ in the measure. These different Lévy processes intuitively correspond to the different ways according to which momentum is transferred to the test particle as a consequence of interaction with the environment. Thus for example a Poisson process corresponds to a situation in which the different possible interaction events are characterized by a fixed momentum transfer, given by the height of the jumps in the Poisson process. More generally a physically realistic situation involves a compound Poisson process, characterized by the fact that the momentum transfer in the single interaction events is not a deterministic quantity, but it is itself described by a probability density, depending on the detail of the microscopic interaction mechanism, according to which the Poisson process is composed.

The function $\Phi\left(t, \boldsymbol{X}-\boldsymbol{Y}\right)$ is a characteristic function, so that it has the following interesting properties, explaining why Eq. (8) generally gives a well defined master equation describing loss of coherence in the position representation:

- $\Phi\left(t, 0\right) = 1$
- $\left|\Phi\left(t, \boldsymbol{X}-\boldsymbol{Y}\right)\right| \leqslant 1$
- $\Phi\left(t, \boldsymbol{X}-\boldsymbol{Y}\right)$ is positive definite
- $\Phi\left(t, \boldsymbol{X}-\boldsymbol{Y}\right) \longrightarrow 0$ for $t \to \infty$
- $\Phi\left(t, \boldsymbol{X}-\boldsymbol{Y}\right) \longrightarrow 0$ for $\left(\boldsymbol{X}-\boldsymbol{Y}\right) \to \infty$, provided there exists a probability density.

These properties typical of characteristic functions[21] automatically entail that the diagonal matrix elements in the position representation are not affected with elapsing time, thus preserving normalization of the statistical operator, while the off-diagonal matrix elements are generally suppressed as expected due to decoherence. Furthermore for a fixed spatial distance $\boldsymbol{X}-\boldsymbol{Y}$ the off-diagonal matrix elements in the position representation are fully suppressed for long enough interaction times, while for a fixed interaction time these off-diagonal matrix elements only go to zero if the associated process admits a proper probability density, which is not the case e.g. for a compound Poisson process. Depending on the particular process describing the random momentum transfers in each scattering event different characteristic functions appear, corresponding to different behaviors in the suppression of the off-diagonal matrix elements for large spatial separations. The function $\left|\Phi\left(t, \boldsymbol{X}-\boldsymbol{Y}\right)\right|$, which is responsible for the loss of visibility in interferometric experiments testing decoherence, for a fixed interaction time t might monotonically decrease to zero for growing values of $\boldsymbol{X}-\boldsymbol{Y}$, or also oscillate and reach asymptotically a finite value corresponding to a residual coherence. These quite different behaviors, corresponding to a more or less effective decoherence effect, are all encoded in the possible expressions of the characteristic function Φ. Application of this formalism to actually realized experiments has been considered in[10]. Typical experiments testing decoherence in a quantitative way involve an interferometer for massive particles (such as fullerenes[22,23] or atoms[24,25]), in which the interfering particle is exposed to some environment during the time of flight, such as a background gas, a laser field or even the internal degrees of freedom of the interfering particle itself.

4. Decoherence in momentum and position for a massive tracer particle

The general structure of translation-covariant quantum dynamical semigroups allows for the description of decoherence effects provided one considers the behavior in time of the so called coherences, that is to say the off-diagonal matrix elements of the statistical operator in a given basis, selected by the dynamics itself or by the observation which can be performed on the open system. For the considered massive particle interacting with some environment the natural basis are given by momentum or position. In order to describe both phenomena we obviously cannot neglect the momentum dynamics as implicitly done going over from Eq. (2) to Eq. (3) and Eq. (4). We therefore need a physical example of realization of the general structure Eq. (2), as given by the quantum version of the classical linear Boltzmann equation[8,26–29]. Such a master equation describes the dynamics of a quantum test particle interacting through collisions with a homogeneous gas, thus providing a quantum counterpart of the classical linear Boltzmann equation. For the case of a scattering cross section $\sigma(\boldsymbol{Q})$ only depending on the momentum transfer the equation can be written

$$\begin{aligned}\mathcal{L}[\rho] = \frac{n_{\text{gas}}}{m_*^2} \int d\boldsymbol{Q} \sigma(\boldsymbol{Q}) \Big[& \mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \sqrt{S(\boldsymbol{Q}, \mathsf{P})} \rho \sqrt{S(\boldsymbol{Q}, \mathsf{P})} \mathrm{e}^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \\ & -\frac{1}{2} \left\{ S(\boldsymbol{Q}, \mathsf{P}), \rho \right\} \Big],\end{aligned} \tag{10}$$

with n_{gas} the density of gas particles with mass m, M the mass of the test particle, $m_* = mM/(m+M)$ the reduced mass, $S(\boldsymbol{Q}, \mathsf{P})$ a two-point correlation function of the gas known as dynamic structure factor and explicitly given by

$$S(\boldsymbol{Q}, \boldsymbol{P}) = \sqrt{\frac{\beta m}{2\pi}} \frac{1}{Q} \exp\left(-\frac{\beta}{8m} \frac{(Q^2 + 2mE(\boldsymbol{Q}, \boldsymbol{P}))^2}{Q^2} \right), \tag{11}$$

with

$$E\left(\boldsymbol{Q},\boldsymbol{P}\right)=\frac{\left(\boldsymbol{P}+\boldsymbol{Q}\right)^{2}}{2M}-\frac{P^{2}}{2M}=\frac{Q^{2}}{2M}+\frac{\boldsymbol{Q}\cdot\boldsymbol{P}}{M} \tag{12}$$

the energy transfer in the single collision and $\beta=1/\left(k_{\mathrm{B}}T\right)$. We are not going to delve on details of the structure of such an equation. We only point out that it actually provides an example of translation-covariant master equation complying with the general mathematical result. We are however interested to show that such a structure actually describes decoherence phenomena in both momentum and position. In fact while the classical linear Boltzmann equation only describes dissipative effects, corresponding to the behavior of populations in momentum space, that is the diagonal matrix elements in the momentum representation of Eq. (10), the quantum master equation also describes coherences and therefore possibly interference phenomena and suppression thereof as a consequence of the dynamics, provided suitable quantum states given by linear superpositions states are considered.

Looking at coherence in momentum space implies considering coherent superpositions of momentum eigenstates. Such highly non classical motional states can show interference effects which are expected to be suppressed as a consequence of the interaction with the environment. As a consequence matrix elements of the form $\langle\boldsymbol{P}|\rho|\boldsymbol{P}'\rangle$ are quickly suppressed for $\boldsymbol{P}\neq\boldsymbol{P}'$, so that for long enough times the dynamics only affects the behavior of the probability density $\langle\boldsymbol{P}|\rho|\boldsymbol{P}\rangle$, and the master equation Eq. (10) goes effectively over to a classical rate equation for such a probability density. Due to the complexity of Eq. (10) obtaining an analytical solution is hardly feasible, so that the natural strategy is to numerically solve the master equation, relying on a so called unraveling of the master equation itself[15], to be solved by means of Monte Carlo methods. In this case setting

$$V\left(\boldsymbol{Q}\right)=\mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar}\sqrt{\frac{n_{\mathrm{gas}}}{m_{*}^{2}}\sigma\left(\boldsymbol{Q}\right)S\left(\boldsymbol{Q},\mathsf{P}\right)}, \tag{13}$$

one can consider the following stochastic differential equation for the stochastic wave vector $\psi\left(t\right)$

$$\begin{aligned}\mathrm{d}|\psi\left(t\right)\rangle=&\left[-\frac{1}{2}\int\mathrm{d}\boldsymbol{Q}V^{\dagger}\left(\boldsymbol{Q}\right)V\left(\boldsymbol{Q}\right)+\frac{1}{2}\int\mathrm{d}\boldsymbol{Q}\|V\left(\boldsymbol{Q}\right)|\psi\left(t\right)\rangle\|^{2}\right]|\psi\left(t\right)\rangle\mathrm{dt}\\&+\int\mathrm{d}\boldsymbol{Q}\left[\frac{V\left(\boldsymbol{Q}\right)|\psi\left(t\right)\rangle}{\|V\left(\boldsymbol{Q}\right)|\psi\left(t\right)\rangle\|}-|\psi\left(t\right)\rangle\right]\mathrm{dN}_{\boldsymbol{Q}}\left(t\right),\end{aligned} \tag{14}$$

where the field of increments satisfies

$$\mathrm{dN}_{\boldsymbol{Q}}(t)\,\mathrm{dN}_{\boldsymbol{Q}'}(t) = \delta^3\left(\boldsymbol{Q}-\boldsymbol{Q}'\right)\mathrm{dN}_{\boldsymbol{Q}}(t)$$
$$\mathbb{E}\left[\mathrm{dN}_{\boldsymbol{Q}}(t)\right] = \|V(\boldsymbol{Q})\,|\psi(t)\rangle\|^2\mathrm{dt},$$

so that indeed the solutions of the stochastic differential equation (14) provide unravelings of the master equation Eq. (10), in the sense that

$$\rho(t) = \mathbb{E}\left[|\psi(t)\rangle\langle\psi(t)|\right].$$

Despite the formal complexity of Eq. (14), for initial states given by momentum eigenvectors one can develop a simple algorithm to study the dynamics of such states, essentially corresponding to the Gillespie algorithm[30], leading to a pure jump process in momentum space. On similar grounds one can also study the dynamics of coherent superpositions of the form

$$|\psi(0)\rangle = \alpha_1(0)\,|\boldsymbol{P}_1\rangle + \alpha_2(0)\,|\boldsymbol{P}_2\rangle,$$

with $\sum_{i=1}^{2}|\alpha_i(0)|^2 = 1$, which evolve in time according to

$$|\psi(t)\rangle = \alpha_1(t)\,|\boldsymbol{P}_1(t)\rangle + \alpha_2(t)\,|\boldsymbol{P}_2(t)\rangle,$$

where again $\sum_{i=1}^{2}|\alpha_i(t)|^2 = 1$. An estimate of loss of coherence can be obtained studying the quantity

$$C(t) = \mathbb{E}\left[\frac{|\alpha_1(t)\,\alpha_2^\star(t)|}{|\alpha_1(0)\,\alpha_2^\star(0)|}\right].$$

As it turns out this measure for the coherence of the state in the momentum basis behaves for a constant scattering cross section approximately as[31]

$$C(t) = \exp\left[-\gamma\left(|\boldsymbol{P}_1-\boldsymbol{P}_2|\right)t\right], \tag{15}$$

where the argument of the exponential is given by

$$\gamma(P) = \Lambda(P) - \Lambda_0\frac{\mathrm{erf}(P)}{P},$$

with

$$\Lambda(P) = \frac{n_{\mathrm{gas}}}{m_*^2}\int \mathrm{d}\boldsymbol{Q}\sigma S(\boldsymbol{Q},\boldsymbol{P}), \tag{16}$$

$\mathrm{erf}(x) = 2\pi^{-\frac{1}{2}}\int_0^x \exp\left(-t^2\right)\mathrm{d}t$ denotes the error function, and Λ_0 is a reference scattering rate given by $\Lambda_0 = n_{\mathrm{gas}}v_{\mathrm{mp}}4\pi\sigma$, with v_{mp} the most probable velocity for the gas particles. Eq. (15) clearly predicts an exponential loss of coherence in the momentum basis, depending on the relative distance in momentum space of the states making up the coherent superposition.

For the study of decoherence in position space we can follow a different strategy. Neglecting in Eq. (10) the dynamics of the momentum, we can replace the corresponding operator by a classical label $\boldsymbol{P}_0$ giving the mean value of the momentum of the incoming particle. The master equation then reads

$$\mathcal{L}[\rho] = \frac{n_{\text{gas}}}{m_*^2} \int \mathrm{d}\boldsymbol{Q} \sigma(\boldsymbol{Q}) S(\boldsymbol{Q}, \boldsymbol{P}_0) \left[\mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \rho \mathrm{e}^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} - \rho \right], \qquad (17)$$

corresponding to a particular realization of Eq. (4). Considering a constant scattering cross section and defining the rate $\Lambda(P_0)$ according to (16) one can introduce the following characteristic function

$$\Phi_S(\boldsymbol{X}) = \frac{n_{\text{gas}}\sigma}{m_*^2 \Lambda(P_0)} \int \mathrm{d}\boldsymbol{Q} S(\boldsymbol{Q}, \boldsymbol{P}_0) \mathrm{e}^{i\boldsymbol{Q}\cdot\boldsymbol{X}/\hbar},$$

so that the master equation (17) can be solved in the position representation as in (8), leading to

$$\langle \boldsymbol{X}|\rho_t|\boldsymbol{Y}\rangle = \exp\left(-\Lambda_0 \frac{2}{\sqrt{\pi}} \left[1 - \Phi_S(\boldsymbol{X} - \boldsymbol{Y})\right] t \right) \langle \boldsymbol{X}|\rho_0|\boldsymbol{Y}\rangle, \qquad (18)$$

where according to the general framework presented in Sect. 3 the characteristic function of a compound Poisson process appears. A suitable measure of decoherence is given in this case by

$$D(t) = \frac{\langle \boldsymbol{X}|\rho_t|\boldsymbol{Y}\rangle}{\langle \boldsymbol{X}|\rho_0|\boldsymbol{Y}\rangle}.$$

For a test particle slower than the gas particles, so that $P_0 \ll M v_{\text{mp}}$, one has

$$\Phi_S(\boldsymbol{X}) \approx {}_1F_1\left(1, \frac{3}{2}; -4\pi \frac{X^2}{\lambda_{\text{th}}^2}\right),$$

with λ_{th} the thermal de Broglie wavelength of the gas particles given by $\lambda_{\text{th}} = \sqrt{2\pi\beta\hbar^2/m}$, and ${}_1F_1$ the confluent hypergeometric function, so that

$$D(t) = \exp\left(-\Lambda_0 \frac{2}{\sqrt{\pi}} \left[1 - {}_1F_1\left(1, \frac{3}{2}; -4\pi \frac{X^2}{\lambda_{\text{th}}^2}\right)\right] t \right),$$

which for spatial distances above the thermal de Broglie wavelength $X \gg \lambda_{\text{th}}$ is well approximated by a fixed decoherence rate $D(t) = \exp(-2\Lambda_0 t/\sqrt{\pi})$, expressing the fact that for large enough distances off-diagonal matrix elements in the position representation are uniformly suppressed.

5. Conclusions and outlook

We have given a brief presentation of how quantum dynamical semigroups can be useful for the description of decoherence in quantum mechanics, as also pursued in[32,33], coping in a quantitative way with experimentally realizable situations. This has been obtained relying on a characterization of translation-covariant quantum dynamical semigroups, leading to a quantum non-commutative generalization of the Lévy-Khintchine formula. When applied to the study of decoherence, neglecting dissipative phenomena, such a structure leads to a description of loss of coherence with a wide variety of possible behaviors, each corresponding to the characteristic function of a classical Lévy process. Despite pursued within the framework of the Markov assumption, thus supposing that the dynamics does not entail memory effects, the approach to the description of decoherence building on covariance properties, recently also followed in[34], can be of more general validity, as it appears from recent results pointing to a generalization of the Lindblad structure for the description of a class of non-Markovian evolutions[35].

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