Covariant Mappings for the Description of Measurement, Dissipation and Decoherence in Quantum Mechanics

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The general formalism of quantum mechanics for the description of statistical experiments is briefly reviewed, introducing in particular position and momentum observables as POVM characterized by their covariance properties with respect to the isochronous Galilei group. Mappings describing state transformations both as a consequence of measurement and of dynamical evolution for a closed or open system are considered with respect to the general constraints they have to obey and their covariance properties with respect to symmetry groups. In particular different master equations are analyzed in view of the related symmetry group, recalling the general structure of mappings covariant under the same group. This is done for the damped harmonic oscillator, the two-level system, and quantum Brownian motion. Special attention is devoted to the general structure of translation-covariant master equations. Within this framework a recently obtained quantum counterpart of the classical linear Boltzmann equation is considered, as well as a general theoretical framework for the description of different decoherence experiments, pointing to a connection between different possible behaviors in the description of decoherence and the characteristic functions of classical Lévy processes.

1 Introduction

Since its very beginning quantum mechanics has urged physicists and other scientists, getting interested in or involved with it, to radically change their classical picture of reality, as well as their way to describe and understand experiments. Despite the elapsed time, the matured knowledge about quantum mechanics and the growing number of applications, the process of deeper understanding of quantum mechanics and of its truly basic features is still on its way. In these notes we will stress the standpoint that quantum mechanics actually is a probability theory, enlarging and modifying the horizons of the classical one and allowing to describe quantitatively

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experiments at the microscopic level. The probabilistic standpoint aiming at an understanding of the statistical structure of quantum theory has in fact turned out to be of great significance in recent achievements in the description of quantum mechanical systems, leading to the introduction of new relevant concepts and tools often in analogy with classical probability theory. In particular we will build on the growing impact that the modern formulation of quantum mechanics in terms of statistical operators, POVM and instruments is having in recent experiments and theoretical studies, as well as the relevance of the characterization of mappings giving the dynamical evolution of both closed and open systems. This applies as well to one-step transformations describing the overall effect of a measurement, as to measurements resolved in their time duration, or more generally irreversible evolutions taking place as a consequence of the interaction of the system of interest with some external system, typically even though not necessarily having many degrees of freedom. The presentation will pay particular attention to the structural features of such mappings and especially to their covariance with respect to the representation of a symmetry group, whenever this applies.

This contribution grew out of the lectures given at the Summer School in Theoretical Physics in Durban and is organized as follows. Section 2 supports the point of view according to which quantum mechanics actually is a probability theory and its general formulation is presented in this spirit, starting from the description of statistical experiments. States are introduced as preparation procedures mathematically represented by statistical operators, observables as registration procedures are described in terms of POVM, and the statistics of experimental outcomes is given in terms of the trace formula. Finally, general state transformations as a consequence of measurement are associated with instruments, which provide the transformed state as well as the statistics of the measurement. Examples are provided focussing on position observables, as well as joint position and momentum observables, understood as specified according to their covariance properties with respect to the isochronous Galilei group. Section 3 explores the relevance of the concept of mapping acting on a space of operators in a given Hilbert space for situations ranging from free evolutions to open system dynamics. Dynamical mappings are characterized in view of possible constraints depending on the physical situation of interest and leading to important informations on their possible structure. In particular we consider the notion of covariance with respect to the representation of a symmetry group, complete positivity and semigroup composition law, corresponding to a Markov approximation. This leads to the Lindblad characterization of generators of quantum-dynamical semigroups, possibly also including covariance requirements. Particular examples of master equation are given stressing their covariance properties with respect to the proper symmetry, also providing the general characterization of master equations covariant under the same group. This is done for the damped harmonic oscillator and shift-covariance, a two-level system and rotation-covariance, quantum Brownian motion and translation-covariance. Finally it is shown how the general expression of a translation-covariant generator, building on a quantum non-commutative version of the Lévy-Khintchine formula, actually encompasses a quantum version of the classical linear Boltzmann equation for the description of the motion of a quantum test particle in a gas, as well as a unified theoretical framework for the explanation of different decoherence experiments.

2 Quantum Mechanics as Quantum Probability

In the present section we will briefly introduce the basic tools necessary in order to describe in the most general way a quantum mechanical system and the possible measurements that can be performed on it. The basic idea that we would like to convey or at least draw to the reader's attention is that quantum mechanics is indeed naturally to be seen as a probability theory, significantly different from the classical one, rather than an extension of classical mechanics. Experiments at the microscopic level are of statistical nature in an essential way and their quantitative description asks for a probabilistic model which is the quantum one, emerged in the twenties and first thoroughly analyzed by von Neumann [1], actually before the foundations of classical probability theory were laid down by Kolmogorov in the thirties [2]. The fact that quantum mechanics is a probability theory different from the classical one, containing the latter as a special case, brings with itself that quantum experiments and their statistical description exhibit new features, which sometimes appear unnatural or paradoxical when somehow forced to fit in a classical probabilistic picture of reality, which is closer to our intuition. Our presentation is more akin to the introduction to quantum mechanics one finds in textbooks on quantum information and communication theory rather than standard quantum mechanics textbooks, even though at variance with the former we will mainly draw examples from systems described in an infinite-dimensional Hilbert space. The standpoint according to which quantum mechanics actually is a probability theory is by now well understood, and even though it is still not in the spirit of typical textbook presentations, it has been developed and thoroughly investigated in various books and monographes (see, e.g., [3–10]), to which we refer the reader for more rigorous and detailed presentations. A more concise account of similar ideas has also been given in [11].

2.1 Classical Statistical Description

The basic setting of classical probability theory as clarified by Kolmogorov is described within the mathematical framework of measure theory. A classical probability model is fixed by specifying a measure space, which is the space of elementary events, a σ-algebra on this measure space characterizing the meaningful events to which we want to ascribe probabilities, and a probability measure on it. The observable quantities are then given by real measurable functions on this space, i.e., random variables. Take for example the case of the classical description of the dynamics of a point particle in three-dimensional space. Then the measure space is given by the usual phase-space $\mathbb{R}^3 \times \mathbb{R}^3$ endowed with the Borel σ-algebra, where the points of phase-space can be identified with position and momentum of the particle. The probability measure can be expressed by means of a probability density $f(\mathbf{x}, \mathbf{p})$, i.e., a positive and normalized element of $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and observables are described as random variables given by real functions $X(\mathbf{x}, \mathbf{p})$ in $L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, so that exploiting the canonical duality relation between L^1 and L^∞ mean values are given by

$$\langle X \rangle_f = \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3\mathbf{x} d^3\mathbf{p}\, X(\mathbf{x}, \mathbf{p}) f(\mathbf{x}, \mathbf{p}).$$

The very same probability density $f(\mathbf{x}, \mathbf{p})$ allows to calculate the expectation value of any random variable, i.e., of any observable. In particular any observable taking values in $\mathbb{R}$ defines a probability measure on this space according to the formula.

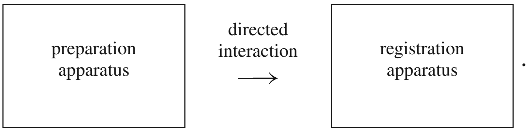
$$\mu^X(M) = \int_{X^{-1}(M)} d^3\mathbf{x} d^3\mathbf{p}\, f(\mathbf{x}, \mathbf{p}),$$

where M is a Borel set in the outcome space $\mathbb{R}$ of the observable and again the same probability density $f(\mathbf{x}, \mathbf{p})$ appears.

As a special case for the position observable $X(\mathbf{x}, \mathbf{p}) = \mathbf{x}$ the measure $\mu^{\mathbf{x}}$ can be expressed by means of the probability density $f^{\mathbf{x}}(\mathbf{x})$ obtained by taking the marginal of $f(\mathbf{x}, \mathbf{p})$ with respect to momentum and similarly for the momentum observable $X(\mathbf{x}, \mathbf{p}) = \mathbf{p}$. In particular one can notice that all observables commute, the points of phase-space can be taken as meaningful elementary events and any probability measure can be uniquely decomposed as a convex mixture of the extreme points of the convex set of probability measures, given by the measures with support concentrated at these elementary events given by single points in phase-space. This probabilistic description is however not mandatory in the classical case, where a deterministic description applies, and it only becomes a very convenient or the only feasible tool for systems with a very high number of degrees of freedom. The situation is quite different in the quantum case.

2.2 Statistics of an Experiment

In the quantum case experiments are by necessity of statistical nature. The most simple setup can typically be described as a suitably devised macroscopic apparatus, possibly made up of lots of smaller components, preparing the microscopical system we would like to study, which in turn triggers another macroscopic device designed to measure the value of a definite quantity. The reproducible quantity to be compared with the theory is the relative frequency according to which the preparation apparatus triggers the registration apparatus in a high enough number of repetitions of the experiment under identical circumstances. A most simple sketch of such a setup can be given by the so-called Ludwig's Kisten [3, 12]



More complicated setups can be traced back to this one by suitably putting together different apparatuses in order to build a new preparation apparatus, and similarly for the registration part. In order to describe such experiments one has to introduce a suitable probability theory, which can actually account for the various experimental evidences to be gained at microscopic level. This is accomplished by introducing mathematical objects describing the preparation and the registration, as well as a statistical formula to extract from these two objects the probability densities to be compared with the experimental outcomes.

States as preparation procedures

In the quantum case a preparation procedure is generally described by a statistical operator. Given the Hilbert space $\mathcal{H}$ in which the system one considers has to be described, e.g., $L^2(\mathbb{R}^3)$, for the center of mass degrees of freedom of a particle in three-dimensional space, statistical operators are positive trace class operators on $\mathcal{H}$ with trace equal to one:

$$\rho \in \mathcal{K}(\mathcal{H}) = \{\rho \in \mathcal{T}(\mathcal{H}) | \, \rho = \rho^\dagger, \rho \geq 0, \operatorname{Tr} \rho = 1\}.$$

The set $\mathcal{K}(\mathcal{H})$ of statistical operators is a convex subset of the space $\mathcal{T}(\mathcal{H})$ of trace class operators on the Hilbert space $\mathcal{H}$, so that any convex mixture of statistical operators is again a statistical operator:

$$\rho_1, \rho_2 \in \mathcal{K}(\mathcal{H}) \Rightarrow w = \mu\rho_1 + (1-\mu)\rho_2 \in \mathcal{K}(\mathcal{H}) \, \text{for} \, 0 \leq \mu \leq 1.$$

In particular the extreme points of such a set are given by one-dimensional projections, that is to say pure states, which cannot be expressed as a proper mixture. We stress the fact that statistical operators are actually to be associated to the considered statistical preparation procedure, rather than to the system itself. More precisely they describe a whole equivalence class of preparation procedures which all prepare the system in the same state, even though by means of quite different macroscopic apparatuses. This correspondence between statistical operators and equivalence classes of preparation procedures is reflected in the fact that a statistical operator generally admits infinitely many different decompositions, as mixtures of pure states or other statistical operators. Relying on the spectral theorem a statistical operator can always be written in the form:

$$\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|, \quad \lambda_j \geq 0, \quad \sum_j \lambda_j = 1,$$

with $\{|\psi_j\rangle\}$ as an orthonormal set. However infinite many others not necessarily orthogonal demixtures generally exist. Think for example of the most simple case of a statistical operator describing the spin state of a fully unpolarized beam of spin $1/2$ particles: $\rho = \frac{1}{2}\mathbb{1}$. Then any orthogonal basis in $\mathbb{C}^2$ (e.g., the eigenvectors of the spin operator along a given arbitrary direction) provides an orthogonal decomposition of the considered statistical operator. Such decompositions correspond to different possible macroscopic procedures leading to such a preparation. In the N runs of the statistical experiment the beam is prepared $\frac{N}{2}$ times with spin $+\frac{\hbar}{2}$ along a fixed direction, and $\frac{N}{2}$ times with spin $-\frac{\hbar}{2}$. All such preparations, differing in the choice of direction, lead to the same state, but they cannot be performed together since no apparatus can measure the spin along two different directions: they are therefore incompatible. The prepared state is however the same and the actual preparation cannot be distinguished on the basis of any other subsequent statistical experiment whatsoever performed on the obtained state. At variance with the classical case, therefore, states are given by operators which generally admit infinitely many convex decompositions and represent equivalence class of preparation procedures.

Observables as registration procedures

On the same footing one has to associate a mathematical object to a macroscopic apparatus assembled in order to measure the value of a certain quantity. Once again utterly different and generally incompatible macroscopic procedures and apparatuses can possibly be used to assign a value to the same physical quantity. The operator describing an observable is therefore to be understood as the mathematical representative of a whole equivalence class of registration procedures. In full generality an observable in the sense clarified above is given by a positive operator-valued measure (POVM), the measure theoretic aspect appearing since one is in fact interested in the probability that the quantity of interest lies within a certain interval.

A POVM is a mapping defined on a suitable measure space and taking values in the set of positive operators within $\mathcal{B}(\mathcal{H})$, that is to say the Banach space of bounded operators on $\mathcal{H}$. Taking for the sake of concreteness an observable assuming values in $\mathbb{R}^3$, such as the position of a particle in three-dimensional space, a POVM is given by a mapping F defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^3)$:

$$\begin{aligned} F : \mathcal{B}(\mathbb{R}^3) &\to \mathcal{B}(\mathcal{H}), \\ M &\to F(M) \end{aligned}$$

associating to each interval $M \in \mathcal{B}(\mathbb{R}^3)$ a positive bounded operator in such a way that

$$\begin{aligned} &0 \leq F(M) \leq \mathbb{1}, \\ &F(\emptyset) = 0, \quad F(\mathbb{R}^3) = \mathbb{1}, \\ &F(\cup_i M_i) = \sum_i F(M_i)\, if\; M_i \cap M_j = \emptyset \,\text{when}\, i \neq j, \end{aligned}$$

where the first condition will turn out to be necessary for the statistical interpretation, the second one expresses normalization, always associating the null operator to the empty set and the identity to the whole space, while the last condition amounts to σ-additivity. For a fixed set $M \in \mathcal{B}(\mathbb{R}^3)$ the operator $F(M)$, positive and between zero and one, is called *effect*. Note that observables are given here by generally noncommuting operators. Moreover we have not requested $F(M)$ to be a projection operator, that is to say a self-adjoint and idempotent operator such that $F^2(M) = F(M)$. If this further condition holds for all $M \in \mathcal{B}(\mathbb{R}^3)$ one has a very special case of POVM, also called projection-valued measure (PVM), since it is a measure taking values in the space of projections on the Hilbert space $\mathcal{H}$. For such measures we shall use the symbol $E(M)$. In the case of PVM there is a one-to-one correspondence between the PVM and a uniquely defined self-adjoint operator, thus explaining the standard definition of observable as self-adjoint operator.

The operator associated to the PVM turns out to be a very convenient tool for the calculation of mean values and higher order moments, such as variances. Let us call $E^{\mathbf{A}}$ the PVM for the description of measurements on the quantity $\mathbf{A}$ taking values in $\mathbb{R}^k$. The first moment of the measure

$$\mathbf{A} = \int_{\sigma(\mathbf{A})} \mathbf{x} dE^{\mathbf{A}}(\mathbf{x})$$

actually identifies k commuting self-adjoint operators, the integral being calculated over the support of the measure or equivalently the spectrum of $\mathbf{A}$, and higher moments of the measure can be identified with powers of these operators, according to the functional calculus:

$$\mathbf{A}^n = \int_{\sigma(\mathbf{A})} \mathbf{x}^n dE^{\mathbf{A}}(\mathbf{x}).$$

In particular a whole collection of commuting self-adjoint operators can be obtained considering the integrals of a measurable function g from $\mathbb{R}^k$ to $\mathbb{R}$:

$$g(\mathbf{A}) = \int_{\sigma(\mathbf{A})} g(\mathbf{x}) dE^{\mathbf{A}}(\mathbf{x}),$$

corresponding to measurements of functions of the quantity $\mathbf{A}$. These facts are no longer true for a generic POVM.

Statistics of outcomes

Having introduced statistical operators as general mathematical representatives of a state, in the sense of characterization of preparation apparatuses, and POVM as mathematical representatives of observables, associated to registration apparatuses, we now have to combine states and observables in order to express the probabilities to be compared with the outcomes of an experiment. This is done by considering the duality relation between the spaces of states and observables. As in the classical description the space of observables L^∞ is the dual of the space L^1 of states; here the Banach space of bounded operators is the dual of the space $\mathcal{T}(\mathcal{H})$ of trace class operators to which statistical operators do belong. The duality relation is given by the trace evaluation:

$$\begin{aligned} \mathrm{Tr} : \mathcal{B}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) &\to \mathbb{C}, \\ (X, w) &\to \mathrm{Tr}\, X^\dagger w, \end{aligned}$$

where taking any basis in $\mathcal{H}$, e.g., $\{u_n\}$, the trace can be evaluated as

$$\mathrm{Tr}\, X^\dagger w = \sum_n \langle u_n | X^\dagger w | u_n \rangle,$$

the series being convergent for any bounded operator X and trace class operator w and the result independent of the choice of orthonormal basis. Given a system prepared in the state ρ, the probability that a quantity described by the POVM F takes value in the set M and is given by the statistical formula:

$$\mathrm{Tr}\, \rho F(M). \tag{1}$$

The property of ρ and F ensure that $\mathrm{Tr}\, \rho F(M)$ is indeed a positive number between zero and one, and in particular for every pair ρ and F the mapping

$$\begin{aligned} \mathrm{Tr}\, \rho F(\cdot) : \mathcal{B}(\mathbb{R}^3) &\to [0, 1], \\ M &\to \mathrm{Tr}\, \rho F(M) \end{aligned}$$

is a classical probability measure assigning to each set M the probability $\mathrm{Tr}\, \rho F(M)$ that the outcome of the experiment lies in that set. For a given state to each observable one can therefore associate a classical probability measure; however only commuting observables are described by the same probability measure, to different observables one generally has to associate distinct probability measures. The formula (1) when considered for the particular case of a pure state $|\psi\rangle$ and a PVM E leads to the usual expression

$$\| E(M) \psi \|^2$$

for the evaluation of the statistics of an experiment measuring E once the system has been prepared in the state $|\psi\rangle$. Considering within this framework the usual notion of position and momentum observables one immediately realizes that the related measures can be expressed by means of the two well-distinct probability densities $|\psi(\mathbf{x})|^2$ and $|\tilde{\psi}(\mathbf{p})|^2$, respectively ($\tilde{\psi}$ denoting as usual the Fourier transform). At variance with the classical case there is in general no common probability density allowing to express the probability measure of all observables. There is in fact no sample space of elementary events. Note that for fixed F the mapping $\rho \to \mathrm{Tr}\, \rho F(\cdot)$

is an affine mapping from the convex set $\mathcal{K}(\mathcal{H})$ of statistical operators into the convex set of classical probability measures on $\mathcal{B}(\mathbb{R}^3)$. For a fixed observable this is all we need in order to compare with the experimental outcomes. Since any such affine mapping can be written in this form for a uniquely defined POVM one can come to the conclusion that POVM indeed provide the most general description of the statistics of experimental outcomes compatible with the probabilistic interpretation of quantum mechanics. The statistical formula (1) is the key point where the theory can be compared with experiment and allows us to better understand the meaning of the equivalence classes. Two preparation apparatuses are in the same equivalence class if they produce the very same statistics of outcomes for any observable, and similarly two registration apparatuses are in the same equivalence class if they lead to the same statistics of outcomes for any state.

2.3 Example of POVM for Position and Momentum

Now we want to consider a few examples of POVM and PVM concentrating on position and momentum, showing in particular how symmetry properties can be a very important guiding principle in the determination of meaningful observables, once we leave the correspondence principle focussed on quantum mechanics as a new mechanics with respect to the classical one. As we shall see while for the case of either position or momentum alone POVM are essentially given by a suitable coarse-graining with respect to the usual PVM; if one wants to give statistical predictions for the measurement of both position and momentum together the corresponding observable is given by necessity in terms of a POVM. For a more detailed and mathematically accurate exposition we refer the reader to [7, 13, 14].

Covariant mapping

Let us first start by introducing the notion of a mapping covariant under a given symmetry group G. As we will show this notion is of great interest in many situations, both for the construction of POVM and general dynamical mappings. Consider a measure space $\mathcal{X}$ with the σ-algebra of Borel sets $\mathcal{B}(\mathcal{X})$. Such a space is called a G-space if there exist an action of G on $\mathcal{X}$ defined as a mapping that sends group elements $g \in G$ to transformation mappings μ_g on $\mathcal{X}$ in such a way as to preserve group composition and identity

$$\mu_g \mu_h = \mu_{gh}, \qquad \forall g, h \in G, \qquad \mu_e = 1_{\mathcal{X}},$$

where $1_{\mathcal{X}}$ denotes the identity function on $\mathcal{X}$. If furthermore G acts transitively on $\mathcal{X}$, in the sense that any two point of $\mathcal{X}$ can be mapped one into the other with $\mu_{g'}$ for a suitable $g' \in G$, then $\mathcal{X}$ is called a transitive G-space. Consider for example $\mathcal{X} = \mathbb{R}^3$; then $\mathcal{X}$ is a transitive G-space with respect to the group of translations. The elements of the group are three-dimensional vectors acting in the obvious way on the Borel sets of $\mathbb{R}^3$, i.e., $\mu_{\mathbf{a}} = M + \mathbf{a}$ for all $\mathbf{a} \in \mathbb{R}^3$ and for all $M \in \mathcal{B}(\mathbb{R}^3)$.

Consider as well a unitary representation $U(g)$ of the same group G on a Hilbert space $\mathcal{H}$:

$$|\psi_g\rangle = U(g)|\psi\rangle \text{ for } \psi \in \mathcal{H} \text{ and } g \in G,$$

in terms of which one also has a representation of G on a space $\mathcal{A}(\mathcal{H})$ of operators acting on $\mathcal{H}$:

$$A_g = U^{\dagger}(g)AU(g) \text{ for } A \in \mathcal{A}(\mathcal{H}).$$

A mapping $\mathcal{M}$ defined on $\mathcal{B}(\mathcal{X})$ and taking values in $\mathcal{A}(\mathcal{H})$ is said to be covariant with respect to the symmetry group G provided it commutes with the action of the group in the sense that

$$U^{\dagger}(g)\mathcal{M}(X)U(g) = \mathcal{M}(\mu_{g^{-1}}(X)) \qquad \forall X \in \mathcal{B}(\mathcal{X}) \quad \forall g \in G. \tag{2}$$

A symmetry transformation on the domain of the mapping is mapped into the symmetry transformation corresponding to the same group element on the range of the mapping.

Position observable

As an example of an observable in the sense outlined above we now want to introduce the position observable. Rather than relying on the usual correspondence principle with respect to classical mechanics, we want to give an operational definition of the position observable, fixing its behavior with respect to the action of the relevant symmetry group, which in this case is the isochronous Galilei group, containing translations, rotations and boosts, that is to say velocity transformations. The group acts in the natural way on the Borel sets of $\mathbb{R}^3$, and the covariance equations that we require for an observable to be interpreted as position observable are the following:

$$\begin{aligned} U^{\dagger}(\mathbf{a})F^{\mathbf{x}}(M)U(\mathbf{a}) &= F^{\mathbf{x}}(M-\mathbf{a}) && \forall\, \mathbf{a} \in \mathbb{R}^3, \\ U^{\dagger}(\mathsf{R})F^{\mathbf{x}}(M)U(\mathsf{R}) &= F^{\mathbf{x}}(\mathsf{R}^{-1}M) && \forall\, \mathsf{R} \in SO(3), \\ U^{\dagger}(\mathbf{q})F^{\mathbf{x}}(M)U(\mathbf{q}) &= F^{\mathbf{x}}(M) && \forall\, \mathbf{q} \in \mathbb{R}^3. \end{aligned} \tag{3}$$

The mapping $F^{\mathbf{x}}$ to be interpreted as a position observable has to transform covariantly with respect to translations and rotations as in (2), and to be invariant under a velocity transformation. These equations can also be seen as a requirement on the possible macroscopic apparata possibly performing such a measurement. The apparatus used to test whether the considered system is localized in the translated region $M - \mathbf{a}$ should be in the equivalence class to which the translated apparatus used to test localization in the region M belongs, and similarly for rotations. Localization measurements should instead be unaffected by boost transformations. A solution of these covariance equations, that is to say a POVM complying with (3), is now a position observable. If one looks for such a solution asking moreover that the

POVM be in particular a PVM, the solution is uniquely given by the usual spectral decomposition of the position operator:

$$E^{\mathbf{x}}(M) = \chi_M(\hat{\mathbf{x}}) = \int_M d^3\mathbf{x}|\mathbf{x}\rangle\langle\mathbf{x}|, \tag{4}$$

where χ_M denotes the characteristic function of the set M. The first moment of the spectral measure gives the usual triple of commuting position operators:

$$\hat{\mathbf{x}} = \int_{\mathbb{R}^3} d^3\mathbf{x}\,\mathbf{x}|\mathbf{x}\rangle\langle\mathbf{x}|,$$

whose powers coincide with the higher moments of $E^{\mathbf{x}}$:

$$\hat{\mathbf{x}}^n = \int_{\mathbb{R}^3} d^3\mathbf{x}\,\mathbf{x}^n|\mathbf{x}\rangle\langle\mathbf{x}|.$$

In particular for a given state ρ mean values and variances of the classical probability distribution giving the position distribution can be expressed by means of the operator $\hat{\mathbf{x}}$:

$$\begin{aligned}\mathsf{Mean}(E^{\mathbf{x}}) &= \mathrm{Tr}\,\rho\hat{\mathbf{x}} = \langle\hat{\mathbf{x}}\rangle_\rho,\\ \mathsf{Var}(E^{\mathbf{x}}) &= \mathrm{Tr}\,\rho\hat{\mathbf{x}}^2 - (\mathrm{Tr}\,\rho\hat{\mathbf{x}})^2 = \langle\hat{\mathbf{x}}^2\rangle_\rho - \langle\hat{\mathbf{x}}\rangle_\rho^2.\end{aligned}$$

The pair $(U, E^{\mathbf{x}})$, where U is the unitary representation of the symmetry group, here the isochronous Galilei group, and $E^{\mathbf{x}}$ a PVM covariant under the action of U is called a system of imprimitivity. More generally a solution of (3) as a POVM is obtained as follows. Let us introduce a rotationally invariant probability density $h(\mathbf{x})$:

$$h(\mathbf{x}) \geq 0, \qquad \int d^3\mathbf{x}\,h(\mathbf{x}) = 1, \qquad h(\mathsf{R}\mathbf{x}) = h(\mathbf{x}),$$

with variance given by

$$\mathsf{Var}(h) = \int d^3\mathbf{x}\,\mathbf{x}^2 h(\mathbf{x}).$$

One can then indeed check that the expression

$$F^{\mathbf{x}}(M) = (\chi_M * h)(\hat{\mathbf{x}}) = \int_M d^3\mathbf{y}\int_{\mathbb{R}^3} d^3\mathbf{x}\,h(\mathbf{x}-\mathbf{y})|\mathbf{x}\rangle\langle\mathbf{x}|, \tag{5}$$

where $*$ denotes convolution, actually is a POVM complying with (3), and in fact provides the general solution of (3). The POVM (5) actually is a smeared version of the usual sharp position observable, the probability density $h(\mathbf{x})$ which fixes the

POVM being understood as the actual, finite resolution of the registration apparatus. For any state ρ the first moment of the associated probability density can still be expressed as the mean value of the usual position operator, since $\mathsf{Mean}(h) = 0$:

$$\begin{aligned}\mathsf{Mean}(F^{\mathbf{x}}) &= \int_{\mathbb{R}^3} d^3\mathbf{y} \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathbf{y} h(\mathbf{x}-\mathbf{y})\, \mathrm{Tr}\, \rho |\mathbf{x}\rangle\langle\mathbf{x}| \\ &= \langle\hat{\mathbf{x}}\rangle_\rho .\end{aligned}$$

The second moment however differs:

$$\begin{aligned}\mathsf{Var}(F^{\mathbf{x}}) &= \int_{\mathbb{R}^3} d^3\mathbf{y} \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathbf{y}^2 h(\mathbf{x}-\mathbf{y})\, \mathrm{Tr}\, \rho |\mathbf{x}\rangle\langle\mathbf{x}| \\ &\quad - \left(\int_{\mathbb{R}^3} d^3\mathbf{y} \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathbf{y} h(\mathbf{x}-\mathbf{y})\, \mathrm{Tr}\, \rho |\mathbf{x}\rangle\langle\mathbf{x}| \right)^2 \\ &= \langle\hat{\mathbf{x}}^2\rangle_\rho - \langle\hat{\mathbf{x}}\rangle_\rho^2 + \mathsf{Var}(h) .\end{aligned}$$

It is not anymore expressed only by the mean value of the operator which can be used to evaluate the first moment and by its square. A further contribution $\mathsf{Var}(h)$ appears, which is state independent, and reflects the finite resolution of the equivalence class of apparatuses used for the localization measurement. Note that the usual result is recovered in the limit of a sharply peaked probability density $h(\mathbf{x}) \to \delta^3(\mathbf{x})$. Taking, e.g., a distribution of the form

$$h_\sigma(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2} \right)^{\frac{3}{2}} e^{-\frac{1}{2\sigma^2}\mathbf{x}^2} \xrightarrow{\sigma\to 0} \delta^3(\mathbf{x}),$$

one has that in the limit of an infinite accuracy in the localization measurement of the apparatus exploited the POVM reduces to the standard PVM:

$$\begin{aligned}F^{\mathbf{x}}(M) &= \int_M d^3\mathbf{y} \int_{\mathbb{R}^3} d^3\mathbf{x} \left(\frac{1}{2\pi\sigma^2} \right)^{\frac{3}{2}} e^{-\frac{1}{2\sigma^2}(\mathbf{x}-\mathbf{y})^2} |\mathbf{x}\rangle\langle\mathbf{x}| \\ &\xrightarrow{\sigma\to 0} \int_M d^3\mathbf{y} \int_{\mathbb{R}^3} d^3\mathbf{x}\, \delta^3(\mathbf{x}-\mathbf{y}) |\mathbf{x}\rangle\langle\mathbf{x}| \\ &= \int_M d^3\mathbf{x}\, |\mathbf{x}\rangle\langle\mathbf{x}| .\end{aligned}$$

Analogous results can obviously be obtained for a momentum observable, asking for the corresponding covariance properties.

Position and momentum observable

A more interesting situation appears when considering apparatuses performing both a measurement of the spatial location of a particle as well as of its momentum. As it is well known no observable can be associated to such a measurement in the framework of standard textbook quantum mechanics. Let us consider the covariance equations of such an observable in the more general framework of POVM. A position and momentum observable should be given by a POVM $F^{\mathbf{x},\mathbf{p}}$ defined on $\mathcal{B}(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfying the following covariance equations under the action of translations, rotations and boosts, respectively:

$$\begin{aligned} U^\dagger(\mathbf{a})F^{\mathbf{x},\mathbf{p}}(M \times N)U(\mathbf{a}) &= F^{\mathbf{x},\mathbf{p}}((M-\mathbf{a}) \times N) && \forall\, \mathbf{a} \in \mathbb{R}^3, \\ U^\dagger(\mathsf{R})F^{\mathbf{x},\mathbf{p}}(M \times N)U(\mathsf{R}) &= F^{\mathbf{x},\mathbf{p}}(\mathsf{R}^{-1}M \times \mathsf{R}^{-1}N) && \forall\, \mathsf{R} \in SO(3), \\ U^\dagger(\mathbf{q})F^{\mathbf{x},\mathbf{p}}(M \times N)U(\mathbf{q}) &= F^{\mathbf{x},\mathbf{p}}(M \times (N-\mathbf{q})) && \forall\, \mathbf{q} \in \mathbb{R}^3. \end{aligned} \tag{6}$$

Such covariance equations, defining a position and momentum observable by means of its operational meaning, do not admit any solution within the set of PVM, while the general solution within the set of POVM is given by

$$F^{\mathbf{x},\mathbf{p}}(M \times N) = \frac{1}{(2\pi\hbar)^3}\int_M d^3\mathbf{x} \int_N d^3\mathbf{p}\, W(\mathbf{x},\mathbf{p})SW^\dagger(\mathbf{x},\mathbf{p}), \tag{7}$$

where S is a trace class operator, positive, with trace equal to one and invariant under rotations

$$S \in \mathcal{T}(\mathcal{H}), \quad S \geq 0, \quad \operatorname{Tr} S = 1, \qquad U^\dagger(\mathsf{R})SU(\mathsf{R}) = S,$$

so that it is in fact a statistical operator, even though it does not have the meaning of a state, while the unitaries

$$W(\mathbf{x},\mathbf{p}) = e^{-\frac{i}{\hbar}(\mathbf{x}\cdot\hat{\mathbf{p}} - \hat{\mathbf{x}}\cdot\mathbf{p})}$$

are the Weyl operators built in terms of the canonical position and momentum operators. The covariance of (7) under (6) can be directly checked, together with its normalization, working with the matrix elements of the operator expression. The couple $(U, F^{\mathbf{x},\mathbf{p}})$, where U is the unitary representation of the symmetry group and $F^{\mathbf{x},\mathbf{p}}$ a POVM covariant under its action, is now called system of covariance. The connection with position and momentum observables as well as the reason why such a joint observable can be expressed only in the formalism of POVM, where position observables alone are generally given by smeared versions of the usual position observable, and similarly for momentum, can be understood looking at the marginal observables. Starting from (7) one can in fact consider a measure of position irrespective of the momentum of the particle, thus coming to the marginal position observable:

$$\begin{aligned} F^{\mathbf{x}}(M) &= F^{\mathbf{x},\mathbf{p}}(M \times \mathbb{R}^3) \\ &= \int_M d^3\mathbf{y} \int_{\mathbb{R}^3} d^3\mathbf{x}\, |\mathbf{x}\rangle\langle \mathbf{x}-\mathbf{y}|S|\mathbf{x}-\mathbf{y}\rangle\langle\mathbf{x}| \\ &= \int_M d^3\mathbf{y} \int_{\mathbb{R}^3} d^3\mathbf{x}\, h_{S^{\mathbf{x}}}(\mathbf{x}-\mathbf{y})|\mathbf{x}\rangle\langle\mathbf{x}|, \end{aligned}$$

where the function

$$h_{S^{\mathbf{x}}}(\mathbf{x}) = \langle\mathbf{x}|S|\mathbf{x}\rangle \tag{8}$$

is a well-defined probability density due to the fact that the operator S has all the properties of a statistical operator, so that $\langle\mathbf{x}|S|\mathbf{x}\rangle$ would be the position probability density of a system described by the state S. On similar grounds the marginal momentum observable is given by

$$\begin{aligned} F^{\mathbf{p}}(N) &= F^{\mathbf{x},\mathbf{p}}(\mathbb{R}^3 \times N) \\ &= \int_N d^3\mathbf{q} \int_{\mathbb{R}^3} d^3\mathbf{p}\, |\mathbf{p}\rangle\langle \mathbf{p}-\mathbf{q}|S|\mathbf{p}-\mathbf{q}\rangle\langle\mathbf{p}| \\ &= \int_N d^3\mathbf{q} \int_{\mathbb{R}^3} d^3\mathbf{p}\, h_{S^{\mathbf{p}}}(\mathbf{p}-\mathbf{q})|\mathbf{p}\rangle\langle\mathbf{p}|, \end{aligned}$$

where again the function

$$h_{S^{\mathbf{p}}}(\mathbf{p}) = \langle\mathbf{p}|S|\mathbf{p}\rangle \tag{9}$$

is a well-defined probability density, which corresponds to the momentum probability density of a system described by the statistical operator S. As it appears the marginal observables are given by two POVM characterized by a smearing of the standard position and momentum observables by means of the probability densities $h_{S^{\mathbf{x}}}(\mathbf{x})$ and $h_{S^{\mathbf{p}}}(\mathbf{p})$, respectively. It is exactly this finite resolution in the measurement of both position and momentum, with two probability densities satisfying

$$\mathsf{Var}_i(h_{S^{\mathbf{x}}})\mathsf{Var}_i(h_{S^{\mathbf{p}}}) \geq \frac{\hbar^2}{4} \text{ for } i = x, y, z$$

as follows from (8) and (9), that allows for a joint measurement for position and momentum in quantum mechanics, without violating Heisenberg's uncertainty relations. In order to consider a definite example we take S to be a pure state corresponding to a Gaussian of width σ:

$$\langle\mathbf{x}|\psi\rangle = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{3}{4}} e^{-\frac{1}{4\sigma^2}\mathbf{x}^2},$$

on which the Weyl operators act as a translation in both position and momentum, leading to

$$\langle \mathbf{x}|W(\mathbf{x}_0, \mathbf{p}_0)|\psi\rangle = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{3}{4}} e^{-\frac{1}{4\sigma^2}(\mathbf{x}-\mathbf{x}_0)^2+\frac{i}{\hbar}\mathbf{p}_0\cdot(\mathbf{x}-\mathbf{x}_0)} = \langle \mathbf{x}|\psi_{\mathbf{x}_0,\mathbf{p}_0}\rangle.$$

In particular one has

$$h_{\psi^{\mathbf{x}}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{3}{2}} e^{-\frac{1}{2\sigma^2}\mathbf{x}^2}, \qquad \mathsf{Var}_i(h_{\psi^{\mathbf{x}}}) = \sigma^2 \qquad \text{for } i = x, y, z$$

and

$$h_{\psi^{\mathbf{p}}}(\mathbf{p}) = \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{3}{2}} e^{-\frac{2\sigma^2}{\hbar^2}\mathbf{p}^2}, \qquad \mathsf{Var}_i(h_{\psi^{\mathbf{p}}}) = \frac{\hbar^2}{4\sigma^2} \qquad \text{for } i = x, y, z$$

so that

$$\mathsf{Var}_i(h_{\psi^{\mathbf{x}}})\mathsf{Var}_i(h_{\psi^{\mathbf{p}}}) = \frac{\hbar^2}{4} \qquad i = x, y, z.$$

The POVM now reads

$$F^{\mathbf{x},\mathbf{p}}(M \times N) = \frac{1}{(2\pi\hbar)^3} \int_M d^3\mathbf{x}_0 \int_N d^3\mathbf{p}_0\, |\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle\langle\psi_{\mathbf{x}_0\mathbf{p}_0}| \tag{10}$$

with marginals

$$F^{\mathbf{x}}(M) = \int_M d^3\mathbf{x}_0 \int_{\mathbb{R}^3} d^3\mathbf{x} \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{3}{2}} e^{-\frac{1}{2\sigma^2}(\mathbf{x}-\mathbf{x}_0)^2} |\mathbf{x}\rangle\langle\mathbf{x}|$$

and

$$F^{\mathbf{p}}(N) = \int_N d^3\mathbf{p}_0 \int_{\mathbb{R}^3} d^3\mathbf{p} \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{3}{2}} e^{-\frac{2\sigma^2}{\hbar^2}(\mathbf{p}-\mathbf{p}_0)^2} |\mathbf{p}\rangle\langle\mathbf{p}|$$

for position and momentum, respectively. It is now clear that depending on the value of σ one can have more or less coarse-grained position and momentum observables. No limit on σ can however be taken in order to have a sharp observable for both position and momentum. In the limit $\sigma \to 0$ one has as before $F^{\mathbf{x}} \to E^{\mathbf{x}}$, but the marginal for momentum would identically vanish, intuitively corresponding to a complete lack of information on momentum, and vice versa.

As a last remark we would like to stress that despite the fact that PVM are only a very particular case of POVM, corresponding to most accurate measurements, self-adjoint operators, which are in one-to-one correspondence to PVM, do play a distinguished and very important role in quantum mechanics as generators of symmetry transformations, according to Stone's theorem.

2.4 Measurements and State Transformations

The formalism presented up to now allows to describe in the most general way the state corresponding to a given preparation procedure and the statistics of the experimental outcomes obtained by feeding a certain registration procedure by such a state. It is however also of interest to have information not only on the statistics of the outcomes, which amounts to provide a classical probability density, but also to specify the state obtained as a consequence of such a measurement, provided the system does not simply get absorbed. This makes it possible to deal, e.g., with repeated consecutive measurements, allowing for a description of continual measurement in quantum mechanics [15], as well as to use the combination of initial preparation and registration apparata altogether as a new preparation apparatus, preparing states according to the value of a certain observable.

State transformations with a measuring character as instruments

The mathematical object characterizing a state transformation as a consequence of a given measurement is called instrument and is generally given by a mapping defined on the outcome space of the measurement, e.g., $\mathcal{B}(\mathbb{R}^3)$ in the examples we have considered, and taking values in the space of bounded mappings acting on the space of trace class operators, obeying the following requirements:

$$\begin{aligned}\mathcal{F}(\cdot):\mathcal{B}(\mathbb{R}^3)&\rightarrow\mathcal{B}(\mathcal{T}(\mathcal{H})),\\ M&\rightarrow\mathcal{F}(M),\end{aligned}$$

$$\begin{aligned}&\mathrm{Tr}\,\mathcal{F}(\mathbb{R}^3)[\rho]=\mathrm{Tr}\,\rho,\\ &\mathcal{F}(\cup_i M_i)=\sum_i\mathcal{F}(M_i)\,\text{if}\,M_i\cap M_j=\emptyset\,\text{for}\,i\neq j\,,\end{aligned}$$

where for each $M\in\mathcal{B}(\mathbb{R}^3)$ the mapping $\mathcal{F}(M)$ is completely positive and generally trace decreasing, transforming trace class operators into trace class operators. $\mathcal{F}(M)$ is often called operation, and the mapping $\mathcal{F}$ is normalized in the sense that $\mathcal{F}(\mathbb{R}^3)$ is trace preserving. As for any completely positive mapping for fixed M one has for $\mathcal{F}(M)$ a Kraus representation:

$$\mathcal{F}(M)[\rho]=\sum_i V_i\rho V_i^{\dagger}.$$

The trace class operator

$$\mathcal{F}(M)[\rho_{\text{in}}]$$

whose trace is generally less than one, describes the subcollection of systems obtained by asking the outcome of the measurement to be in $M \in \mathcal{B}(\mathbb{R}^3)$. After the measurement in fact the transformed system can be sorted according to the outcome of the measurement itself. The transformed state according to a measurement without readout, i.e., without making any selection with respect to the result of the measurement (the so-called a priori state) is given by

$$\rho_{\text{out}} = \mathcal{F}(\mathbb{R}^3)[\rho_{\text{in}}],$$

the mapping $\mathcal{F}(\mathbb{R}^3)$ now only describing the modification on the incoming state ρ_{in} as a consequence of its interaction with the registration apparatus. The transformed state conditioned on the result of the measurement, i.e., the new state obtained by sorting out only the systems for which the outcome of the measurement was in $M \in \mathcal{B}(\mathbb{R}^3)$ (the so-called a posteriori state) is given by

$$\rho_{\text{out}}(M) = \frac{\mathcal{F}(M)[\rho_{\text{in}}]}{\operatorname{Tr} \mathcal{F}(M)[\rho_{\text{in}}]}.$$

Of course an instrument does not only provide the transformed trace class operator describing the system after its interaction with the registration apparatus performing the measurement, but also the statistics of the outcomes. The probability of an outcome $M \in \mathcal{B}(\mathbb{R}^3)$ for a measurement described by the instrument $\mathcal{F}$ on an incoming state ρ is given by the formula:

$$\operatorname{Tr} \mathcal{F}(M)[\rho],$$

which can also be expressed by means of a POVM F uniquely determined by the instrument $\mathcal{F}$ as follows:

$$F(M) = \mathcal{F}'(M)[\mathbb{1}],$$

where $\mathcal{F}'$ denotes the adjoint mapping with respect to the trace evaluation

$$\operatorname{Tr} \mathcal{F}(M)[\rho] = \operatorname{Tr} \mathbb{1}(\mathcal{F}(M)[\rho]) = \operatorname{Tr}(\mathcal{F}'(M)[\mathbb{1}])\rho = \operatorname{Tr} F(M)[\rho].$$

Note that the correspondence between instruments and POVM is not one-to-one. In fact there are different registration apparatuses, possibly leading to quite different transformations on the incoming state, which however all provide a measurement of the same observable. As a consequence while an instrument uniquely defines the associated POVM as outlined above, generally infinitely many different instruments are compatible with a given POVM, corresponding to different macroscopic implementation of measurements of the same observable. Note further that if the mapping $\mathcal{F}$ is reversible it is necessarily given by a unitary transformation and therefore does not have any measuring character, the system transforms in a reversible way because of its free evolution described by a self-adjoint operator. A very special case of instrument can be obtained starting from the knowledge of an observable given as self-adjoint operator, $A = \sum_i a_i E_i$, where $\{E_i\}$ is a collection of mutually orthogonal projection operators, summing up to the identity, $E_i = E_i^2$ and $\sum_i E_i = \mathbb{1}$. Then the mapping

$$\mathcal{F}(M)[\rho] = \sum_{\{i|a_i \in M\}} E_i \rho E_i$$

actually is an instrument describing the state transformation of an incoming state ρ as a consequence of the measurement of the observable A as predicted by von Neumann's projection postulate. If the experimenter detects the value a_i for A the transformed state is given by

$$\mathcal{F}(\{a_i\})[\rho] = E_i \rho E_i .$$

This instrument has the peculiar property of being repeatable, in the sense that from

$$\mathcal{F}(M)[\mathcal{F}(N)[\rho]] = \sum_{\{i|a_i \in M \cap N\}} E_i \rho E_i ,$$

it follows

$$\mathcal{F}(\{a_i\})[\mathcal{F}(\{a_i\})[\rho]] = \mathcal{F}(\{a_i\})[\rho] = E_i \rho E_i ,$$

that is to say subsequent measurements of the same observable do always lead to the same result, which implicitly means an absolute precision in the measurement of the observable. As it appears this is a very particular situation, which can only be realized for a measurement in the sense of PVM of an observable with discrete spectrum.

Example of instrument for position and momentum

We now provide an example of an instrument corresponding to the description of a state transformation taking place by jointly measuring position and momentum of a particle in $L^2(\mathbb{R}^3)$, whose uniquely associated POVM is just the one given in (10). Consider in fact the mapping

$$\mathcal{F}^{\mathbf{x},\mathbf{p}}(M \times N)[\rho] = \frac{1}{(2\pi\hbar)^3} \int_M d^3\mathbf{x}_0 \int_N d^3\mathbf{p}_0 \, |\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle\langle\psi_{\mathbf{x}_0\mathbf{p}_0}|\rho|\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle\langle\psi_{\mathbf{x}_0\mathbf{p}_0}| \, ,$$

built in terms of the normalized Gaussian wave packets $|\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle$ centered in $(\mathbf{x}_0, \mathbf{p}_0)$. This mapping depends on the interval $M \times N$ as a σ-additive measure, thanks to the fact that it is expressed by means of an operator density with respect to the Lebesgue measure. Normalization is ensured by the completeness relation for Gaussian wave packets

$$\frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3} d^3\mathbf{x}_0 \int_{\mathbb{R}^3} d^3\mathbf{p}_0 \, |\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle\langle\psi_{\mathbf{x}_0\mathbf{p}_0}| = \mathbb{1},$$

leading to

$$\begin{aligned}\operatorname{Tr}\mathcal{F}^{\mathbf{x},\mathbf{p}}(\mathbb{R}^3\times\mathbb{R}^3)[\rho] &= \frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3} d^3\mathbf{x}_0 \int_{\mathbb{R}^3} d^3\mathbf{p}_0 \, \langle\psi_{\mathbf{x}_0\mathbf{p}_0}|\rho|\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle, \\ &= \operatorname{Tr}\rho,\end{aligned}$$

while the adjoint mapping acting on the identity operator

$$\begin{aligned}\mathcal{F}^{\mathbf{x},\mathbf{p}\prime}(M\times N)[\mathbb{1}] &= \frac{1}{(2\pi\hbar)^3}\int_{M} d^3\mathbf{x}_0 \int_{N} d^3\mathbf{p}_0 \, |\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle\langle\psi_{\mathbf{x}_0\mathbf{p}_0}|\mathbb{1}|\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle\langle\psi_{\mathbf{x}_0\mathbf{p}_0}| \\ &= \frac{1}{(2\pi\hbar)^3}\int_{M} d^3\mathbf{x}_0 \int_{N} d^3\mathbf{p}_0 \, |\psi_{\mathbf{x}_0\mathbf{p}_0}\rangle\langle\psi_{\mathbf{x}_0\mathbf{p}_0}| \\ &= F^{\mathbf{x},\mathbf{p}}(M\times N)\end{aligned}$$

immediately gives the joint position and momentum POVM considered in (10). More generally, one can consider an instrument of the form:

$$\begin{aligned}\mathcal{F}^{\mathbf{x},\mathbf{p}}(M\times N) = \frac{1}{(2\pi\hbar)^3}&\int_{M} d^3\mathbf{x}_0 \int_{N} d^3\mathbf{p}_0 \\ &\times W(\mathbf{x}_0,\mathbf{p}_0)\sqrt{S}W^\dagger(\mathbf{x}_0,\mathbf{p}_0)\rho W(\mathbf{x}_0,\mathbf{p}_0)\sqrt{S}W^\dagger(\mathbf{x}_0,\mathbf{p}_0),\end{aligned}$$

which is again well defined due to the relation

$$\frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^3} d^3\mathbf{x}_0 \int_{\mathbb{R}^3} d^3\mathbf{p}_0 \, W(\mathbf{x}_0,\mathbf{p}_0)SW^\dagger(\mathbf{x}_0,\mathbf{p}_0) = \mathbb{1},$$

where S is a positive operator given by a statistical operator invariant under rotations, and whose adjoint mapping applied to the identity

$$\begin{aligned}\mathcal{F}^{\mathbf{x},\mathbf{p}\prime}(M\times N)[\mathbb{1}] &= \frac{1}{(2\pi\hbar)^3}\int_{M} d^3\mathbf{x}_0 \int_{N} d^3\mathbf{p}_0 \, W(\mathbf{x}_0,\mathbf{p}_0)SW^\dagger(\mathbf{x}_0,\mathbf{p}_0) \\ &= F^{\mathbf{x},\mathbf{p}}(M\times N)\end{aligned}$$

coincides with the general expression for a covariant position and momentum POVM observable given in (7).

3 Open Systems and Covariance

In the previous section we have outlined the modern formulation of quantum mechanics, understood as a probability theory necessary for the description of the outcomes of statistical experiments involving microscopic systems. In this framework pure states are generally replaced by statistical operators, observables in the sense of self-adjoint operators are substituted by mappings taking values in an operator space, von Neumann's projection postulate is a very special case of mappings with a measuring character describing state transformations as a consequence of measurement. As it appears the notion of mapping taking values in an operator space becomes very natural and of great relevance. The knowledge of an instrument for example allows not only to predict the statistics of the outcomes of a certain measurement, but also the transformation of the state due to the interaction with the measuring apparatus. On the contrary the spontaneous transformation of the state of a closed system due to passing time is described by a very special kind of mappings, unitary time evolutions uniquely determined by fixing a self-adjoint operator. Considering more generally an open system, that is to say a system interacting with some other external system, its irreversible evolution in time as a consequence of this interaction is determined by a suitable mapping, whose characterization is a very intricate and interesting subject, together with the possibility of understanding and describing such an evolution as a measurement effected on the system. While it is relatively easy to state the general properties that should be obeyed by such mappings in order to provide a well-defined time evolution, the characterization of the structure of such mappings in its full generality is an overwhelmingly complicated problem. Important and quite general results can however be obtained considering constraints coming from physical or mathematical considerations. Also in this section we will aim at a brief introduction of key concepts, skipping all mathematical details, referring the reader to [16, 8, 7, 9, 17] for a more exhaustive presentation.

3.1 Constraints on Dynamical Mappings

In Sect. 2 we have already mentioned two important constraints applying to mappings describing how a state transforms in time, both as a consequence of a measurement or of its dynamical evolution. Actually the two situations are not of a completely different nature, even though in the first case the time extension of the interaction between system and measuring apparatus is typically though not necessarily assumed to be very short and neglected, so that the whole transformation is considered as a one-step process. In the case of the evolution of a closed or open system on the contrary the explicit time dependence of the mapping is essential, while a decomposition of the mapping according to the measurement outcome for a certain observable is generally not available.

Complete positivity

The first constraint was the by now well-known requirement of complete positivity of the mapping. It is a mathematical condition, which at the beginning was somewhat mistrusted by physicists, naturally coming in the foreground in quantum mechanics because of the tensor product structure of the space in which to describe composite systems, thus playing an important role in the theory of entanglement. A completely positive mapping is a mapping which remains positive when extended in a trivial way, i.e., by taking the tensor product with the identity mapping, on a composite Hilbert space. Considering a positive mapping $\mathcal{M}$ in the Schrödinger picture, acting on the space of states $\mathcal{T}(\mathcal{H})$:

$$\begin{aligned} \mathcal{M} : \mathcal{T}(\mathcal{H}) &\rightarrow \mathcal{T}(\mathcal{H}) \\ \rho &\rightarrow \mathcal{M}[\rho], \end{aligned}$$

complete positivity amounts to the requirement that the mapping

$$\begin{aligned} \mathcal{M}_n : \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^n) &\rightarrow \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^n) \\ \rho \otimes \sigma_n &\rightarrow \mathcal{M}[\rho] \otimes \sigma_n \end{aligned}$$

is positive for any $n \in \mathbb{N}$, with σ_n a statistical operator in $\mathbb{C}^n$. An equivalent requirement can be formulated on the adjoint mapping $\mathcal{M}'$ in Heisenberg picture acting on the space of observables $\mathcal{B}(\mathcal{H})$

$$\begin{aligned} \mathcal{M}' : \mathcal{B}(\mathcal{H}) &\rightarrow \mathcal{B}(\mathcal{H}), \\ B &\rightarrow \mathcal{M}'[B]. \end{aligned}$$

$\mathcal{M}'$ is completely positive provided

$$\sum_{i,j=1}^{n} \langle \psi_i | \mathcal{M}'(B_i^\dagger B_j) \psi_j \rangle \geq 0 \text{ for } \{\psi_i\} \subset \mathcal{H} \text{ and } \{B_i\} \subset \mathcal{B}(\mathcal{H})$$

for any $n \in \mathbb{N}$. As it was shown by Kraus, any completely positive mapping can be expressed as follows:

$$\mathcal{M}[\rho] = \sum_i V_i \rho V_i^\dagger$$

with a suitable collection $\{V_i\}$ of operators also called Kraus operators, as already mentioned in Sect. 2.4. As it appears from this fundamental result, of great significance in applications, the condition of complete positivity is quite restrictive, thus allowing for important characterizations.

Covariance

Another important constraint, this time however only arising in the presence of symmetries, is given by the requirement of covariance, already considered in Sec. 2.3. For the case of a mapping defined in an operator space, e.g., in the Schrödinger picture sending statistical operators to statistical operators, given a unitary representation $U(g)$ of the group G on $\mathcal{H}$, the requirement of covariance can be expressed as follows:

$$\mathcal{M}[U(g)\rho U^{\dagger}(g)] = U(g)\mathcal{M}[\rho]U^{\dagger}(g) \qquad \forall g \in G. \tag{11}$$

This condition expresses the fact that the action of the mapping and of the representation of G on $\mathcal{T}(\mathcal{H})$ commute, and automatically implies the same property for the adjoint mapping $\mathcal{M}'$ acting on the space of observables $\mathcal{B}(\mathcal{H})$. Such a condition typically applies when a symmetry, available in the system one is studying, is not spoiled by the transformations brought about on the system by letting it interact with another system, be it a reservoir or a measuring apparatus. The possibility of giving a general solution of the covariance equation (11) obviously depends on the unitary representation of the group and on the class of mappings considered, possibly giving very detailed information on the general structure of such mappings.

Semigroup evolution

For the case of a closed system we know that the mapping giving the reversible time evolution is a one-parameter group of unitary transformations, fixed by a self-adjoint Hamiltonian according to Stone's theorem. A broader class of time evolutions allowing for an irreversible dynamics can be obtained by relaxing the group property to a semigroup composition law, corresponding to the existence of a preferred time direction. In particular one can introduce a so-called quantum-dynamical semigroup, which is a collection of one-parameter mappings $\{\mathcal{U}_t\}_{t\in\mathbb{R}_+}$ such that

$$\begin{aligned} \mathcal{U}_t : \mathcal{T}(\mathcal{H}) &\rightarrow \mathcal{T}(\mathcal{H}), \\ \rho &\rightarrow \mathcal{U}_t[\rho], \end{aligned}$$

is completely positive and trace preserving for any $t \geq 0$, for $t = 0$ one has the identity mapping, and the following semigroup composition law applies:

$$\mathcal{U}_t = \mathcal{U}_{t-s}\mathcal{U}_s \qquad \forall t \geq s \geq 0. \tag{12}$$

The semigroup condition (12) is sometimes called Markov condition, because it expresses the fact that the time evolution of the system does not exhibit memory effects, in analogy to the notion of Markov semigroup in classical probability theory. It tells us that the evolution up to time t can be obtained by arbitrarily composing the evolution mapping up to an intermediate time s with an evolution mapping depending only on the residual time $t - s$ acting on the state $\rho_s = \mathcal{U}_s[\rho_0]$, not referring to the knowledge of the state at previous times $\{\rho_{t'}\}_{0\leq t'\leq s}$. The requirement of complete positivity for this family of mappings allows for a most important characterization of the so-called generator $\mathcal{L}$ of the quantum-dynamical semigroup, which is the mapping giving the infinitesimal time evolution, defined through the relation:

$$\mathcal{U}_t = e^{t\mathcal{L}}.$$

According to a celebrated result of Gorini, Kossakowski, Sudarshan and Lindblad [18, 19] of enormous relevance in the applications, the general structure of the generator of a quantum-dynamical semigroup is given in the Schrödinger picture by

$$\mathcal{L}[\rho] = -\frac{\mathrm{i}}{\hbar}[H, \rho] + \sum_j [L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\}], \tag{13}$$

where H is a self-adjoint operator, and the operators L_j are often called Lindblad operators. The expression (13) is also called a master equation, since it provides the infinitesimal time evolution of the statistical operator, according to $d\rho/dt = \mathcal{L}[\rho]$. The key point is now obviously to determine the explicit expression of H and L_j relevant for the reduced dynamics of the physical system of interest, typically depending on the external reservoir and the details of the interaction mechanism. As we shall see further restrictions on $\mathcal{L}$ can arise as a consequence of an available symmetry not destroyed by the interaction. Note that introducing the operator

$$K = \frac{\mathrm{i}}{\hbar}H + \frac{1}{2}\sum_j L_j^\dagger L_j,$$

where the effective Hamiltonian H appears together with an operator $\frac{\hbar}{2}\sum_j L_j^\dagger L_j$ which can be formally seen as an imaginary, optical effective potential, the Lindblad structure (13) can also be written as

$$\mathcal{L}[\rho] = -K\rho - \rho K^\dagger + \sum_j L_j \rho L_j^\dagger,$$

leading to a Dyson expansion of the semigroup evolution

$$\begin{aligned} \mathcal{U}_t[\rho] &= e^{t\mathcal{L}}[\rho] = \mathcal{K}_t[\rho] \\ &+ \sum_{n=1}^{\infty} \int \cdots \int_{0 \le t_1 \le \ldots \le t_n \le t} dt_1 \ldots dt_n \, \mathcal{K}_{t_1}[\mathcal{J}[\mathcal{K}_{t_2 - t_1} \ldots \mathcal{J}[\mathcal{K}_{t - t_n}[\rho]] \ldots]], \end{aligned} \tag{14}$$

where the superoperators

$$\mathcal{J}[\rho] = \sum_j L_j \rho L_j^\dagger \qquad \mathcal{K}_t[\rho] = e^{-Kt} \rho e^{-K^\dagger t}$$

appear. The formal solution of the time evolution (14) has a quite intuitive physical meaning; in fact it can be seen as a sequence of relaxing evolutions over a time interval t given by the contraction semigroup $\mathcal{K}_t$ interrupted by jumps described by the completely positive mapping $\mathcal{J}$. The sum of over all possible such evolutions gives the final state. Most recently a general characterization has been obtained also for a class of non-Markovian time evolutions which provides a kind of generalization of the Lindblad result (13), in the sense that the state evolved up to a given time can be expressed as a mixture of subcollections (that is positive trace class operators with trace less than one) each obeying a Lindblad type of master equation, however, with different Lindblad operators [20–24], so that the overall time evolution does no more obey a semigroup composition law as in (12).

3.2 Shift-Covariance and Damped Harmonic Oscillator

As a first example of a master equation corresponding to a completely positive quantum-dynamical semigroup let us consider the well-known master equation for a damped harmonic oscillator [25], describing, e.g., the damping of an electromagnetic field mode in a cavity. The Hilbert space of the single mode is given by $\mathcal{H} = l^2(\mathbb{C})$, the square summable sequences over the complex field, with basis $\{|n\rangle\}$, the ket $|n\rangle$ denoting as usual the eigenvector with eigenvalue $n \in \mathbb{N}$ of the number operator $N = a^\dagger a$, $a^\dagger$ and a being respectively creation and annihilation operators of a photon in the given mode of frequency ω. The master equation then reads

$$\begin{aligned}\mathcal{L}_{\mathrm{DHO}}[\rho] = &-\frac{i}{\hbar}[H_0(N), \rho] + \eta(N_\beta(\omega) + 1)[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}] \\ &+\eta N_\beta(\omega)[a^\dagger \rho a - \frac{1}{2}\{aa^\dagger, \rho\}],\end{aligned} \tag{15}$$

where $N_\beta(\omega)$ denotes the average of the photon number operator over a thermal distribution

$$N_\beta(\omega) = \frac{1}{e^{\beta\hbar\omega} - 1} = \frac{1}{2}[\coth(\beta\hbar\omega/2) - 1],$$

$H_0(N) = \hbar\omega N$ is the free Hamiltonian and η the relaxation rate.

Dissipation and decoherence for the damped harmonic oscillator

As it is well known such a master equation describes both classical dissipative effects as well as quantum decoherence effects. To see this let us first focus on dissipative effects, considering the time evolution of the mean amplitude and mean number of quanta in the mode. Considering the adjoint mapping of (15), giving the time evolution in Heisenberg picture

$$\begin{aligned}\mathcal{L}'_{\mathrm{DHO}}[X] = &+\frac{i}{\hbar}[H_0(N), \rho] + \eta(N_\beta(\omega) + 1)[a^\dagger \rho a - \frac{1}{2}\{a^\dagger a, \rho\}] \\ &+\eta N_\beta(\omega)[a\rho a^\dagger - \frac{1}{2}\{aa^\dagger, \rho\}],\end{aligned}$$

one can solve the Heisenberg equations of motion for $X \to a$ and $X \to N = a^\dagger a$, finally obtaining

$$\langle a(t)\rangle = \mathrm{Tr}(a(t)\rho) = \langle a\rangle e^{-\mathrm{i}\omega t-\frac{\eta}{2}t},$$
$$\langle N(t)\rangle = \mathrm{Tr}(N(t)\rho) = \langle N\rangle e^{-\eta t} + N_\beta(\omega)(1-e^{-\eta t}),$$

where $a(t)$, $N(t)$ denote Heisenberg operators at time t, with $a = a(0)$, $N = N(0)$ the corresponding Schrödinger operators. The mean amplitude of the mode thus rotates in the complex plane, vanishing for long enough times, with a decay rate given by $(\eta/2)^{-1}$; the population of the mode goes from the initial value to a final thermal distribution with a decay rate given by η^{-1}. For the study of decoherence we shall consider the time evolution of an initial state given by a coherent superposition of two coherent states characterized by two amplitudes α and β [9]. Setting

$$\rho_0 = \frac{1}{\mathcal{N}_0}[|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + |\alpha\rangle\langle\beta| + h.c.],$$

one can look for the time evolved state, exploiting the fact that coherent states remain coherent states under time evolution. Working for simplicity at zero temperature, so that $N_\beta(\omega) = 0$ one has

$$\rho_t = \frac{1}{\mathcal{N}_t}[|\alpha(t)\rangle\langle\alpha(t)| + |\beta(t)\rangle\langle\beta(t)| + e^{-\frac{1}{2}|\alpha-\beta|^2(1-e^{-\eta t})}|\alpha(t)\rangle\langle\beta(t)| + h.c.],$$

where $\alpha(t) = \alpha e^{-i\omega t-\frac{\eta}{2}t}$ and similarly for $\beta(t)$. As it immediately appears, the so-called coherences, that is to say the off-diagonal matrix elements of the statistical operator, are suppressed with respect to diagonal ones by a factor that for long enough times is given by

$$e^{-\frac{1}{2}|\alpha-\beta|^2} = |\langle\alpha|\beta\rangle|,$$

that is to say the modulus of the overlap of the two coherent states, a tiny quantity for two macroscopically distinguishable states of the electromagnetic field.

Structure of the mapping and covariance

It is immediately apparent that the master equation (15) is an example of a realization of the Lindblad structure (13) with just two Lindblad operators given by

$$L_1 = \sqrt{\eta(N_\beta(\omega)+1)}a, \qquad L_2 = \sqrt{\eta N_\beta(\omega)}a^\dagger,$$

with Hamiltonian

$$H_0(N) = \hbar\omega N$$

and stationary solution

$$w \propto e^{-\beta H_0(N)}. \tag{16}$$

More than this it is also covariant under the action of the group $U(1)$. Consider in fact the unitary representation of $U(1)$ on $\mathcal{H} = l^2(\mathbb{C})$ given by

$$U(\theta) = e^{\mathrm{i}\theta N} \text{ with } \theta \in [0, 2\pi],$$

where N is the usual number operator. The master equation (15) is covariant under this unitary representation of the group $U(1)$ according to

$$\mathcal{L}_{\mathrm{DHO}}[U(\theta)\rho U^{\dagger}(\theta)] = U(\theta)\mathcal{L}_{\mathrm{DHO}}[\rho]U^{\dagger}(\theta) \tag{17}$$

as can be checked immediately. One can also show that this is the unique structure of master equation bilinear in a and $a^{\dagger}$ complying with the Lindblad structure, covariant under $U(1)$ and admitting (16) as a stationary state [26].

It is also possible to give a complete characterization of the structure of the generator of a quantum-dynamical semigroup covariant with respect to $U(1)$ as in (17). The general result has been obtained by Holevo [27, 28] and is given by the following expression:

$$\begin{aligned}
\mathcal{L}[\rho] = -\frac{\mathrm{i}}{\hbar}[H(N), \rho] + \sum_j \left[A_{0j}(N)\rho A_{0j}^{\dagger}(N) - \frac{1}{2}\left\{ A_{0j}^{\dagger}(N)A_{0j}(N), \rho \right\} \right] \\
+ \sum_{m=1}^{\infty}\sum_j \left[W^m A_{mj}(N)\rho A_{mj}^{\dagger}(N)W^{\dagger m} - \frac{1}{2}\left\{ A_{mj}^{\dagger}(N)A_{mj}(N), \rho \right\} \right] \\
+ \sum_{m=1}^{\infty}\sum_j \left[W^{\dagger m} A_{-mj}(N)\rho A_{-mj}^{\dagger}(N)W^m - \frac{1}{2}\left\{ A_{-mj}^{\dagger}(N)P_m A_{-mj}(N), \rho \right\} \right],
\end{aligned}$$

where $A_{mj}(N)$ are functions of the number operator N, the generator of the symmetry; the operator W is given by

$$W = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$$

acting as a shift $|n\rangle \to |n+1\rangle$ on the basis of eigenvectors of the number operator, so that this kind of symmetry is also called shift-covariance, while P_m is the projection on the subspace spanned by $\{|n\rangle\}_{n=m,\ldots,+\infty}$ given by

$$P_m \equiv \sum_{n=m}^{\infty} |n\rangle\langle n| = W^m W^{\dagger m}$$

and one further has

$$U(\theta)W^m = e^{i\theta m} W^m U(\theta), \tag{18}$$

which compared to (28) can be seen as a generalized Weyl relation, expressed by means of the isometric but not unitary operators W^m [29]. Examples of realizations of this general shift-covariant expression are given by the master equation for the damped harmonic oscillator as indicated above, corresponding to the choice

$$A_1(N) = \sqrt{\eta N_\beta(\omega)}\sqrt{N+1}, \qquad A_{-1}(N) = e^{\frac{1}{2}\beta\hbar\omega}\sqrt{\eta N_\beta(\omega)}\sqrt{N},$$
$$A_m(n) = 0 \; m = 0, |m| > 1, \qquad H(N) = \hbar\omega N,$$

as can be checked immediately exploiting the polar representation for the creation and annihilation operators:

$$a = W^\dagger\sqrt{N}, \qquad a^\dagger = W\sqrt{N+1}.$$

A more general structure still preserving the stationary solution (16) is given by the choice

$$A_0(n) = \eta_0, \qquad A_m(N) = \sqrt{\eta_m} N_\beta^{\frac{m}{2}}(\omega)\sqrt{\frac{(N+m)!}{N!}},$$
$$A_{-m}(N) = e^{\frac{m}{2}\beta\hbar\omega}\sqrt{\eta_m} N_\beta^{\frac{m}{2}}(\omega)\sqrt{\frac{N!}{(N-m)!}}, \qquad H(N) = \hbar\omega N,$$

corresponding to

$$\begin{aligned}\mathcal{L}[\rho] = &-\frac{\mathrm{i}}{\hbar}[H_0(N), \rho] - \eta_0[N, [N, \rho]] \\ &+ \sum_{m=1}^{+\infty} \eta_m (N_\beta(\omega)+1)^m [a^m \rho a^{\dagger m} - \frac{1}{2}\{a^{\dagger m} a^m, \rho\}] \\ &+ \sum_{m=1}^{+\infty} \eta_m N_\beta^m(\omega)[a^{\dagger m} \rho a^m - \frac{1}{2}\{a^m a^{\dagger m}, \rho\}],\end{aligned}$$

where a phase damping term is given by a double commutator with the number operator, as well as many photon processes with different decay rates appear.

3.3 Rotation-Covariance and Two-Level System

Another example of a well-known master equation which can be characterized in terms of covariance properties comes from the description of a two-level system interacting with a thermal reservoir, e.g., a two-level atom in the presence of the radiation field or a spin in a magnetic field, so that the Hilbert space is now simply $\mathbb{C}^2$. It corresponds to the so-called Bloch equation and is typically used in quantum optics and magnetic resonance theory. Focussing on a two-level atom with transition frequency ω and spontaneous emission rate η interacting with the quantized electromagnetic field one has

$$\mathcal{L}_{2\mathrm{LS}}[\rho] = -\frac{\mathrm{i}}{\hbar}[H_0(\sigma_z), \rho] + \eta(N_\beta(\omega)+1)[\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}] \\ +\eta N_\beta(\omega)[\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}], \qquad (19)$$

where $N_\beta(\omega)$ is the thermal photon number at the transition frequency and as usual $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y)$ with $\{\sigma_i\}_{i=x,y,z}$, the Pauli matrices.

Dissipation and decoherence for the two-level system

As it is well known this master equation predicts relaxation to a stationary state which is diagonal in the basis of eigenvectors of the free Hamiltonian, with a relative population between ground and excited state determined by the temperature of the bath. This can be immediately seen considering as usual the adjoint mapping of (19):

$$\mathcal{L}'_{2\mathrm{LS}}[X] = +\frac{\mathrm{i}}{\hbar}[H_0(\sigma_z), X] + \eta(N_\beta(\omega)+1)[\sigma_+X\sigma_- - \frac{1}{2}\{\sigma_+\sigma_-, X\}] \\ +\eta N_\beta(\omega)[\sigma_-X\sigma_+ - \frac{1}{2}\{\sigma_-\sigma_+, X\}]$$

and solve the Heisenberg equations of motion for the operator $X \to P_e = |1\rangle\langle 1|$ representing the population in the excited state

$$\langle P_e(t)\rangle = \mathrm{Tr}(P_e(t)\rho) = \langle P_e\rangle e^{-\bar{\eta}t} + \frac{N_\beta(\omega)}{2N_\beta(\omega)+1}(1-e^{-\bar{\eta}t}),$$

which also fixes $\langle P_g(t)\rangle = 1 - \langle P_e(t)\rangle$ due to the normalization condition, where we have denoted as $\bar{\eta} = \eta(2N_\beta(\omega)+1)$ the total transition rate and $P_e = P_e(0)$. Due to the simplicity of the equation also the evolution in time of the coherences is easily determined, now considering the evolution in the Heisenberg picture of the operator $X \to C = |0\rangle\langle 1|$, which is given by

$$\langle C(t)\rangle = \mathrm{Tr}(C(t)\rho) = \langle C\rangle e^{-\mathrm{i}\omega t - \frac{\bar{\eta}}{2}t},$$

where again $C = C(0)$. With elapsing time the populations reach a stationary value, while coherences get suppressed.

pStructure of the mapping and covariance

In strict analogy to the results presented in Sect. 3.2 also the master equation (19) can be immediately recast in Lindblad form (13) with two Lindblad operators given by

$$L_1 = \sqrt{\eta(N_\beta(\omega)+1)}\sigma_-, \qquad L_2 = \sqrt{\eta N_\beta(\omega)}\sigma_+$$

and Hamiltonian

$$H_0(\sigma_z) = \frac{\hbar\omega}{2}\sigma_z\,,$$

admitting the stationary solution

$$w \propto e^{-\beta H_0(\sigma_z)}. \tag{20}$$

The master equation is also covariant under the group $SO(2)$, describing rotations along a given axis and being isomorphic to $U(1)$, which accounts for the similarity between the two equations, even though important differences appear due to the different Hilbert spaces in which the two systems are described and the group represented. Consider in fact the representation of $SO(2)$ in $\mathbb{C}^2$:

$$U(\phi) = e^{\frac{\mathrm{i}}{\hbar}\phi S_z} \qquad \phi \in [0, 2\pi]$$

with $S_z = \frac{\hbar}{2}\sigma_z$: one immediately checks that (19) is covariant in the sense that

$$\mathcal{L}_{2\mathrm{LS}}[U(\phi)\rho U^\dagger(\phi)] = U(\phi)\mathcal{L}_{2\mathrm{LS}}[\rho]U^\dagger(\phi). \tag{21}$$

Also in this case it is possible to provide the explicit expression of the generator of a rotation-covariant quantum-dynamical semigroup in the sense of (21). It takes the simple form [30]:

$$\mathcal{L}[\rho] = -\frac{\mathrm{i}}{\hbar}[H(\sigma_z), \rho] + \sum_{m=0,\pm 1} c_m [T_{1m}\rho T_{1m}^\dagger - \frac{1}{2}\{T_{1m}^\dagger T_{1m}, \rho\}],$$

where the c_m are positive constants and T_{1m} are irreducible tensor operators given by

$$T_{11} = -\frac{1}{\sqrt{2}}(\sigma_x + \mathrm{i}\sigma_y), \qquad T_{10} = \sigma_z, \qquad T_{1-1} = \frac{1}{\sqrt{2}}(\sigma_x - \mathrm{i}\sigma_y)$$

or equivalently in terms of σ_z, σ_+ and σ_-, in order to allow for a direct comparison with (19):

$$\begin{aligned}\mathcal{L}[\rho] = &-\frac{\mathrm{i}}{\hbar}[H(\sigma_z), \rho] - \frac{c_0}{2}[\sigma_z, [\sigma_z, \rho]] \\ &+2c_{-1}[\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}] + 2c_1[\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}].\end{aligned}$$

The use of the word rotation-covariance hints at the fact that such a master equation applies, e.g., to a spin $1/2$ in an environment with axial symmetry, so that one has invariance under rotations along a given axis. More generally the full rotation group $SO(3)$ can be considered, and a general characterization also exists for the structure of generators of semigroups acting on $\mathbb{C}^n$ and invariant under $SO(3)$ [31], relevant for the description of relaxation of a spin j under the influence of isotropic surroundings.

3.4 Translation-Covariance and Quantum Brownian Motion

As a further example of the concepts introduced above we will consider the master equation for the description of quantum Brownian motion, which applies to the motion of a massive test particle in a gas of lighter particles. The Hilbert space of relevance is given here by $L^2(\mathbb{R}^3)$, and denoting the usual position and momentum operators as $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ the general structure of the master equation reads

$$\mathcal{L}_{\mathrm{QBM}}[\rho] = -\frac{\mathrm{i}}{\hbar}[H_0(\hat{\mathbf{p}}), \rho] - \frac{\mathrm{i}}{\hbar}\frac{\eta}{2}\sum_i [\hat{x}_i, \{\hat{p}_i, \rho\}] \\ -\frac{D_{pp}}{\hbar^2}\sum_i [\hat{x}_i, [\hat{x}_i, \rho]] - \frac{D_{xx}}{\hbar^2}\sum_i [\hat{p}_i, [\hat{p}_i, \rho]], \tag{22}$$

where we have assumed isotropy for simplicity, the index $i = x, y, z$ denoting the different Cartesian coordinates and $H_0(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^2/2M$ is the free Hamiltonian. This and similar types of master equation, bilinear in the position and momentum operators, do appear in different contexts, leading to different microscopic expressions for the coefficients, as well as to the appearance of other terms such as a double commutator $[\hat{x}_i, [\hat{p}_i, \rho]]$ with position and momentum operators [9, 32]. We will here have in mind the dynamics of a test particle interacting through collisions with a homogeneous gas [33–35], so that the coefficients read

$$D_{pp} = \eta\frac{M}{\beta}, \qquad D_{xx} = \eta\frac{\beta\hbar^2}{16M},$$

with M the mass of the test particle.

Dissipation and decoherence for quantum Brownian motion

As in the previous case we want to point briefly out how such a master equation describes both dissipative and decoherence effects. Regarding dissipation one has similarly to the classical case that the mean value of momentum is driven to zero, while the average of the squared momentum goes to the equipartition value fixed by the gas temperature. The effect of decoherence typically manifests itself in the fact that superpositions of spatially macroscopically distinguished states are quickly suppressed. Let us first focus on dissipation, considering the adjoint mapping of (22) in Heisenberg picture:

$$\mathcal{L}'_{\rm QBM}[X] = +\frac{\rm i}{\hbar}[H_0(\hat{\mathbf{p}}), X] + \frac{\rm i}{\hbar}\frac{\eta}{2}\sum_i \{\hat{p}_i, [\hat{x}_i, X]\}$$
$$-\frac{D_{pp}}{\hbar^2}\sum_i [\hat{x}_i, [\hat{x}_i, X]] - \frac{D_{xx}}{\hbar^2}\sum_i [\hat{p}_i, [\hat{p}_i, X]].$$

As already mentioned the observables of interest are given by the momentum operators $X \to \hat{\mathbf{p}}$ and the kinetic energy $X \to E = \hat{\mathbf{p}}^2/2M$, whose mean values evolve according to

$$\langle \hat{\mathbf{p}}(t) \rangle = \mathrm{Tr}(\hat{\mathbf{p}}(t)\rho) = \langle \hat{\mathbf{p}} \rangle e^{-\eta t},$$
$$\langle E(t) \rangle = \mathrm{Tr}(E(t)\rho) = \langle E \rangle e^{-2\eta t} + \frac{3}{2\beta}(1 - e^{-2\eta t}),$$

where again $\hat{\mathbf{p}}(t)$ and $E(t)$ denote Heisenberg operators at time t. The average momentum thus relaxes to zero with a decay rate η^{-1}, while the mean kinetic energy reaches the equipartition value with a rate $(2\eta)^{-1}$. We now concentrate on the study of decoherence, both for position and momentum. This can be done considering the off-diagonal matrix elements in both position and momentum of a statistical operator evolved in time according to (22). To do this we exploit the knowledge of the exact solution [36], neglecting however the contribution $-\frac{\rm i}{\hbar}\frac{\eta}{2}\sum_i [\hat{x}_i, \{\hat{p}_i, \rho\}]$ responsible for dissipative effects, that would lead to too a cumbersome expression. Considering an initial state ρ_0 the state up to time t reads in the momentum representation:

$$\langle \mathbf{p}|\rho_t|\mathbf{q}\rangle = e^{-\frac{D_{xx}}{\hbar^2}(\mathbf{p}-\mathbf{q})^2 t} e^{-\frac{1}{12}\frac{D_{pp}}{\hbar^2}\left(\frac{(\mathbf{p}-\mathbf{q})}{M}t\right)^2 t}$$
$$\times \left(\frac{\hbar^2}{4\pi D_{pp} t}\right)^{3/2} \int d^3\mathbf{k}\, e^{-\frac{\hbar^2 \mathbf{k}^2}{4D_{pp}t}} \langle \mathbf{p}-\mathbf{k}|\rho_0|\mathbf{q}-\mathbf{k}\rangle.$$

It appears immediately that off-diagonal matrix elements are quickly suppressed with elapsing time, depending on their separation $(\mathbf{p}-\mathbf{q})^2$. The factor depending on the coefficient D_{xx} is due to the momentum localization term $\sum_i [\hat{p}_i, [\hat{p}_i, \rho]]$, while the factor where the coefficient D_{pp} appears is due to the position localization term $\sum_i [\hat{x}_i, [\hat{x}_i, \rho]]$; this also suppresses coherences in momentum because different momentum states quick-ly lead to spatial separation, so that the position localization mechanism is again of relevance. As far as coherences in position are concerned one has to consider the matrix elements of ρ_t in the position representation. Here it is convenient to express the exact solution ρ_t in terms of the solution ρ_t^S of the free Schrödinger equation. One has the quite cumbersome expression:

$$\langle \mathbf{x}|\rho_t|\mathbf{y}\rangle = e^{-\frac{D_{pp}}{\hbar^2}(\mathbf{x}-\mathbf{y})^2 t\left[1-\frac{D_{pp}}{4M^2}t^2\frac{1}{\left[D_{xx}+\frac{D_{pp}}{3M^2}t^2\right]}\right]}\left(\frac{\hbar^2}{4\pi\left[D_{xx}+\frac{D_{pp}}{3M^2}t^2\right]t}\right)^{3/2}$$

$$\times\int d^3\mathbf{z}\, e^{-\frac{\hbar^2\mathbf{z}^2}{4\left[D_{xx}+\frac{D_{pp}}{3M^2}t^2\right]t}}\, e^{\frac{\mathrm{i}}{\hbar}\frac{D_{pp}}{2M}\frac{\mathbf{z}\cdot(\mathbf{x}-\mathbf{y})t}{\left[D_{xx}+\frac{D_{pp}}{3M^2}t^2\right]}}\langle \mathbf{x}-\mathbf{z}|\rho_t^S|\mathbf{y}-\mathbf{z}\rangle,$$

which is essentially given by a convolution of the free solution with a Gaussian kernel, multiplied by an exponential factor suppressing off-diagonal matrix elements according to their distance $(\mathbf{x}-\mathbf{y})^2$ in space. Spatially macroscopically distant states are again very quickly suppressed.

Structure of the mapping and covariance

The master equation (22) can also be written manifestly in Lindblad form, as it can be seen introducing a single Lindblad operator for each Cartesian direction

$$L_i = \sqrt{\eta}a_i\ ,$$

with

$$a_i = \frac{1}{\sqrt{2}\lambda_{\mathrm{th}}}\left(\hat{x}_i + \frac{\mathrm{i}}{\hbar}\lambda_{th}^2\hat{p}_i\right), \qquad \lambda_{\mathrm{th}} = \sqrt{\frac{\beta\hbar^2}{4M}}, \qquad [a_i, a_j^\dagger] = \delta_{ij}\ ,$$

and the effective Hamiltonian

$$H = H_0(\hat{\mathbf{p}}) + \frac{\eta}{2}\sum_i\{\hat{x}_i, \hat{p}_i\},$$

leading to

$$\mathcal{L}_{\mathrm{QBM}}[\rho] = -\frac{\mathrm{i}}{\hbar}\left[H_0(\hat{\mathbf{p}}) + \frac{\eta}{2}\sum_i\{\hat{x}_i, \hat{p}_i\}, \rho\right] + \eta\sum_i[a_i\rho a_i^\dagger - \frac{1}{2}\{a_i^\dagger a_i, \rho\}]$$

with a stationary solution

$$w \propto e^{-\beta H_0(p)}. \tag{23}$$

Also the quantum Brownian motion master equation is characterized by covariance under the action of a symmetry group which in this case is the group $\mathbb{R}^3$ of translations. Given the unitary representation

$$U(\mathbf{a}) = e^{\frac{\mathrm{i}}{\hbar}\mathbf{a}\cdot\hat{\mathbf{p}}} \quad \text{for}\ \mathbf{a}\in\mathbb{R}^3 \tag{24}$$

of the group of translations on $L^2(\mathbb{R}^3)$ where the operators $\hat{\mathbf{p}}$ act as generator of the symmetry, one can indeed check immediately that (22) is covariant under this representation in the sense that

$$\mathcal{L}_{\mathrm{QBM}}[U(\mathbf{a})\rho U^\dagger(\mathbf{a})] = U(\mathbf{a})\mathcal{L}_{\mathrm{QBM}}[\rho]U^\dagger(\mathbf{a}). \tag{25}$$

In this case however, at odds with the case of the master equation for the damping harmonic oscillator, the three requirements of Lindblad structure, covariance under $\mathbb{R}^3$ and a stationary state given by (23) do not uniquely fix the form of a master equation at most bilinear in the operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ [26].

A most important general characterization of structures of generators of quantum-dynamical semigroups covariant under translations has been obtained by Holevo, relying on a non-commutative quantum generalization [29, 37, 38] of the classical Lévy-Khintchine formula (see e.g. [9, 39]). In this case the generator can be written as follows:

$$\mathcal{L}[\rho] = -\frac{\mathrm{i}}{\hbar}[H(\hat{\mathbf{p}}), \rho] + \mathcal{L}_G[\rho] + \mathcal{L}_P[\rho],$$

where $H(\hat{\mathbf{p}})$ is a self-adjoint operator only depending on the momentum operators, while $\mathcal{L}_G$ and $\mathcal{L}_P$ correspond to a Gaussian and a Poisson component, as in the Lévy–Khintchine formula, and are given by

$$\begin{aligned}\mathcal{L}_G[\rho] = &-\frac{\mathrm{i}}{\hbar}\left[\hat{y}_0 + \frac{1}{2\mathrm{i}}\sum_{k=1}^{r}(\hat{y}_k L_k(\hat{\mathbf{p}}) - L_k^\dagger(\hat{\mathbf{p}})\hat{y}_k), \rho\right] \\ &+\sum_{k=1}^{r}\left[(\hat{y}_k + L_k(\hat{\mathbf{p}}))\rho(\hat{y}_k + L_k(\hat{\mathbf{p}}))^\dagger - \frac{1}{2}\left\{(\hat{y}_k + L_k(\hat{\mathbf{p}}))^\dagger(\hat{y}_k + L_k(\hat{\mathbf{p}})), \rho\right\}\right]\end{aligned} \tag{26}$$

and

$$\begin{aligned}\mathcal{L}_P[\rho] = &\int_{\mathbb{R}^3}\sum_{j=1}^{\infty}\left[U(\mathbf{q})L_j(\mathbf{q},\hat{\mathbf{p}})\rho L_j^\dagger(\mathbf{q},\hat{\mathbf{p}})U^\dagger(\mathbf{q}) - \frac{1}{2}\left\{L_j^\dagger(\mathbf{q},\hat{\mathbf{p}})L_j(\mathbf{q},\hat{\mathbf{p}}), \rho\right\}\right]d\mu(\mathbf{q}) \\ &+\int_{\mathbb{R}^3}\sum_{j=1}^{\infty}\Big[\omega_j(\mathbf{q})(U(\mathbf{q})\rho L_j^\dagger(\mathbf{q},\hat{\mathbf{p}})U^\dagger(\mathbf{q}) - \rho L_j^\dagger(\mathbf{q},\hat{\mathbf{p}})) \\ &+(U(\mathbf{q})L_j(\mathbf{q},\hat{\mathbf{p}})\rho U^\dagger(\mathbf{q}) - L_j(\mathbf{q},\hat{\mathbf{p}})\rho)\omega_j^*(\mathbf{q})\Big]\,d\mu(\mathbf{q}) \\ &+\int_{\mathbb{R}^3}\sum_{j=1}^{\infty}\left[U(\mathbf{q})\rho U^\dagger(\mathbf{q}) - \rho - \frac{\mathrm{i}}{\hbar}\frac{[\mathbf{q}\cdot\hat{\mathbf{x}}, \rho]}{1+|\mathbf{q}|^2}\right]|\omega_j(\mathbf{q})|^2 d\mu(\mathbf{q})\end{aligned} \tag{27}$$

respectively, where

$$\hat{y}_k = \sum_{i=1}^{3} a_{ki} \hat{x}_i, \qquad k = 0, \ldots, r \leq 3, \qquad a_{ki} \in \mathbb{R}$$

are linear combinations of the three position operators $\hat{x}_i$, $L_k(\hat{\mathbf{p}})$ and $L_j(\mathbf{q}, \hat{\mathbf{p}})$ and are generally complex functions of the momentum operators,

$$U(\mathbf{q}) = e^{\mathrm{i}\mathbf{q}\cdot\hat{\mathbf{x}}}$$

are unitary operators corresponding to a translation in momentum or boost, satisfying together with (24) the Weyl form of the canonical commutation rules

$$U(\mathbf{a})U(\mathbf{q}) = e^{\mathrm{i}\mathbf{q}\cdot\mathbf{a}} U(\mathbf{q})U(\mathbf{a}), \tag{28}$$

$\omega_j(\mathbf{q})$ complex functions and $\mu(\mathbf{q})$ a positive measure on $\mathbb{R}^3$ satisfying the Lévy condition

$$\int_{\mathbb{R}^3} \frac{|\mathbf{q}|^2}{1 + |\mathbf{q}|^2} \sum_{j=1}^{\infty} |\omega_j(\mathbf{q})|^2 d\mu(\mathbf{q}) < +\infty.$$

As it appears that this is quite a rich structure, allowing for the description of a very broad class of physical phenomena, only having in common invariance under translations. A first application has already been mentioned above when considering the master equation for quantum Brownian motion, describing the motion of a quantum test particle in a gas, close to thermal equilibrium, which corresponds to a particular realization of the Gaussian component (26). Further examples related to more recent research work will be considered in the next section.

3.5 Translation-Covariant Mappings for the Description of Dissipation and Decoherence

In the previous section we have given the general expression of a translation-covariant generator of a quantum-dynamical semigroup according to Holevo's results. This operator expression can be actually distinguished in to two parts, as also happens in the classical Lévy-Khintchine formula, providing the general characterization of the exponent of the characteristic function of a classical Lévy process (see e.g., [9] for a concise presentation from a physicist's standpoint or [39] for a more thorough probabilistic treatment). Very roughly speaking in the presence of translation-covariance the dynamics can be described essentially in terms of momentum exchanges between test particle and reservoir. The Gaussian part corresponds to a situation in which the dynamics is determined by a very large number of very small momentum transfers, which in the case of finite variance leads to a Gaussian process. For example, in the case of quantum Brownian motion the test particle is close to thermal equilibrium, so that typical values of its momentum are much bigger than that of the gas particles due to its bigger mass, and the momentum changes imparted in single collisions are therefore relatively small. In contrast, the Poisson part can account for a situation in which few interaction events corresponding to significant momentum transfers drive the dynamics, as happens for example in experiments on collisional decoherence, where very few kicks already lead to a significant loss of coherence. In addition to this the general expression can also account for peculiar situations, typical of Lévy processes, where the variance or the mean of the momentum transfers do diverge, so that rare but extremely strong events can give the predominant contributions to the dynamics. We will now try to exemplify such situations referring to recent research work, thus showing how paying attention to covariance properties of dynamical mappings does not only lead to a better and deeper understanding of known results as for the case of damped harmonic oscillator, two-level system and quantum Brownian motion briefly considered in Sects. 3.2, 3.3 and 3.4 respectively, but also provides important insights into the treatment of more complicated problems, allowing for a unified description of apparently quite different situations.

Dissipation and quantum linear Boltzmann equation

The well-known quantum Brownian motion master equation (22) provides, as we have seen, an example of a realization of the Gaussian component (26) of the general structure of generators of translation-covariant quantum-dynamical semigroups specified above. A further example involving the Poisson component (27) can be given, still considering the reduced dynamics of the center of mass of a test particle interacting through collisions with a gas, however, not focussing on the case of a very massive particle close to thermal equilibrium, so that momentum transfers and therefore energy transfers due to collision events between test particle and gas are not necessarily small anymore. This kinetic stage of dynamical description asks for a quantum version of the classical linear Boltzmann equation, the equation being linear in the sense that the gas is supposed to be and remain in equilibrium, so that only the state of the test particle evolves in time. Such a master equation has been obtained recently and its expression in the case in which the scattering cross-section describing the collisions between test particle and gas only depends on the transferred momentum $\mathbf{q}$, which is always true in Born approximation, is given by [33, 40–42]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] + \frac{2\pi}{\hbar}(2\pi\hbar)^3 n \int d^3\mathbf{q}\, |\tilde{t}(\mathbf{q})|^2 \times \Bigg[e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} \sqrt{S(\mathbf{q}, E(\mathbf{q}, \hat{\mathbf{p}}))}\rho\sqrt{S(\mathbf{q}, E(\mathbf{q}, \hat{\mathbf{p}}))} e^{-\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} - \frac{1}{2}\left\{S(\mathbf{q}, E(\mathbf{q}, \hat{\mathbf{p}})), \rho\right\} \Bigg], \tag{29}$$

where n is the gas density, $\tilde{t}(q)$ is the Fourier transform of the interaction potential, S a two-point correlation function known as dynamic structure factor, depending on momentum transfer $\mathbf{q}$ and energy transfer $E(\mathbf{q}, \mathbf{p})$:

$$E(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})^2}{2M} - \frac{\mathbf{p}^2}{2M}.$$

The dynamic structure factor for a free gas of particles obeying Maxwell–Boltzmann statistics has the explicit expression:

$$S_{\mathrm{MB}}(\mathbf{q}, E) = \sqrt{\frac{\beta m}{2\pi}} \frac{1}{q} e^{-\frac{\beta}{8m}\frac{(2mE+q^2)^2}{q^2}},$$

while the general definition of dynamic structure factor reads

$$S(\mathbf{q}, E) = \frac{1}{2\pi\hbar} \int dt \int d^3\mathbf{x}\, e^{\frac{i}{\hbar}(Et-\mathbf{q}\cdot\mathbf{x})} \frac{1}{N} \int d^3\mathbf{y}\ \langle N(\mathbf{y})N(\mathbf{x}+\mathbf{y},t)\rangle ,$$

that is to say it is the Fourier transform with respect to momentum and energy transfer of the two-point density–density correlation function of the medium, $N(\mathbf{y})$ being the number density operator of the gas. This indirectly tells us that the dynamics of the test particle is indeed driven by the density fluctuations in the fluid, due to its discrete microscopic nature [43]. In the structure of the master equation the unitary operators $e^{\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}$ and $e^{-\frac{i}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}$ appearing to the left and the right of the statistical operator do account for the momentum transfer imparted to the test particle as a consequence of a certain collision; the rate with which collisions characterized by a certain momentum transfer happen however, do depend on the actual momentum of the test particle described by the operator $\hat{\mathbf{p}}$ through the dependence of the dynamic structure factor S on $\mathbf{p}$. This mechanism accounts for the approach to equilibrium, favoring collisions driving the kinetic energy of the test particle towards the equipartition value. The result can also be generalized to an arbitrary scattering cross section, not necessarily only depending on momentum transfer, by introducing in the master equation instead of the scattering cross section an operator-valued scattering amplitude, averaged over the gas particles momenta [44]. One can indeed check that the quantum linear Boltzmann equation (29) and its generalizations, apart from being in Lindblad form and manifestly covariant, do admit the correct stationary state (23) and drive the kinetic energy to its equipartition value.

Decoherence and Lévy processes

In the present section we will show how translation-covariant quantum-dynamical semigroups can provide a unified theoretical description of quite different decoherence experiments. At odds with the previous section we are not interested in the dynamics of the momentum observable, decoherence due to spatial localization usually takes place on a much shorter timescale than relaxation phenomena. We therefore neglect in the general expressions (26) and (27) of a translation-covariant generator the dependence on the momentum operator, which we take as a classical label, characterized, e.g., by the mean momentum of the incoming test particle. The formulas (26) and (27) drastically simplify to

$$\mathcal{L}_G[\rho] = -\mathrm{i}\sum_{i=1}^{3} b_i\,[\hat{x}_i, \rho] - \frac{1}{2}\sum_{i,j=1}^{3} D_{ij}\left[\hat{x}_i, \left[\hat{x}_j, \rho\right]\right], \tag{30}$$

$$\mathcal{L}_P[\rho] = \int d\mu(\mathbf{q})|\lambda(\mathbf{q})|^2 \left[e^{\frac{\mathrm{i}}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}}\rho e^{-\frac{\mathrm{i}}{\hbar}\mathbf{q}\cdot\hat{\mathbf{x}}} - \rho - \frac{\mathrm{i}}{\hbar}\frac{[\mathbf{q}\cdot\hat{\mathbf{x}}, \rho]}{1+|\mathbf{q}|^2}\right], \tag{31}$$

with

$$b_i \in \mathbb{R}, \quad D_{ij} \geq 0, \quad |\lambda(\mathbf{q})|^2 = \sum_{j=1}^{\infty} |L_j^{\dagger}(\mathbf{q}) + \omega_j(\mathbf{q})|^2,$$

whose matrix elements in the position representation simply read

$$\langle \mathbf{x}|\mathcal{L}_G[\rho] + \mathcal{L}_P[\rho]|\mathbf{y}\rangle = -\Psi(\mathbf{x}-\mathbf{y})\langle \mathbf{x}|\rho|\mathbf{y}\rangle$$

with

$$\begin{aligned}\Psi(\mathbf{x}) = \mathrm{i}\sum_{i=1}^{3} b_i x_i + \frac{1}{2}\sum_{i=1}^{3} D_{ij} x_i x_j \\ - \int d\mu(\mathbf{q})|\lambda(\mathbf{q})|^2 \left[e^{\frac{\mathrm{i}}{\hbar}\mathbf{q}\cdot\mathbf{x}} - 1 - \frac{\mathrm{i}}{\hbar}\frac{\mathbf{q}\cdot\mathbf{x}}{1+|\mathbf{q}|^2}\right],\end{aligned} \tag{32}$$

which is exactly the general expression of the characteristic exponent of a classical Lévy process [9, 39]. Neglecting the free contribution the equation for the time evolution of the statistical operator in the position representation, which now becomes

$$\frac{d}{dt}\langle \mathbf{x}|\rho|\mathbf{y}\rangle = -\Psi(\mathbf{x}-\mathbf{y})\langle \mathbf{x}|\rho|\mathbf{y}\rangle,$$

has solution

$$\langle \mathbf{x}|\rho_t|\mathbf{y}\rangle = e^{-t\Psi(\mathbf{x}-\mathbf{y})}\langle \mathbf{x}|\rho_0|\mathbf{y}\rangle \equiv \Phi(t, \mathbf{x}-\mathbf{y})\langle \mathbf{x}|\rho_0|\mathbf{y}\rangle, \tag{33}$$

given by multiplying the matrix elements of the initial statistical operator by the characteristic function

$$\Phi(t, \mathbf{x}) = e^{-t\Psi(\mathbf{x})}$$

of a classical Lévy process evaluated at a point $(\mathbf{x} - \mathbf{y})$ given by the difference between the two spatial locations characterizing bra and ket with which the matrix elements of ρ_t are taken. In view of the general properties of the characteristic function Φ, which is the Fourier transform of a probability density, that is to say

$$\Phi(t,0) = 1, \qquad |\Phi(t,\mathbf{x}-\mathbf{y})| \leq 1, \tag{34}$$
$$\Phi(t,\mathbf{x}-\mathbf{y}) \xrightarrow{(\mathbf{x}-\mathbf{y})\to\infty} 0, \qquad \Phi(t,\mathbf{x}-\mathbf{y}) \xrightarrow{t\to\infty} 0,$$

the solution (33) actually predicts on general grounds an exponential loss of coherence in position, that is to say diagonalization in the localization basis. In fact according to (34) diagonal matrix elements are left untouched by the dynamics, which together with the fact that a characteristic function is actually a positive definite function accounts for the correct probability and positivity preserving time evolution. For growing time off-diagonal matrix elements are fully suppressed, whatever the distance, while for fixed time evolution the reduction of off-diagonal matrix elements depends on the relative distance ($\mathbf{x}-\mathbf{y}$), leading to a vanishing contribution for macroscopic distances (provided the corresponding classical Lévy process admits a proper probability density). An application of this general theoretical treatment to actual physical systems and in particular to relevant experimental situations relies on a choice of functions and parameters appearing in (32) dictated by actual physical input. This has been accomplished in [45], where this general scheme has been connected to actual experiments on decoherence, as well as theoretical predictions of decoherence effects when the reservoir inducing decoherence is a quantum chaotic system.

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References

1. J. von Neumann: *Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin 1932) 41
2. A.N. Kolmogorov: *Fundamental Concepts of Probability Theory* (Nauka, Moscow 1974) 41
3. G. Ludwig: *Foundations of Quantum Mechanics. I*, (Springer-Verlag, New York 1983); *Foundations of Quantum Mechanics. II*, (Springer-Verlag, New York 1985) 41, 42
4. A.S. Holevo: *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam 1982) 41
5. K. Kraus: *States, Effects and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin 1983) 41
6. R.F. Streater: J. Math. Phys. **41**(6), 3556–3603 (2000) 41
7. A.S. Holevo: *Statistical Structure of Quantum Theory* (Springer, Berlin 2001) 41, 47, 58
8. R. Alicki and M. Fannes: *Quantum Dynamical Systems* (Oxford University Press, Oxford 2001) 41, 58
9. H.-P. Breuer and F. Petruccione: *The Theory of Open Quantum Systems* (Oxford University Press, Oxford 2007) 41, 58, 63, 68, 71, 72, 75
10. F. Strocchi: *An Introduction to the Mathematical Structure of Quantum Mechanics* (World Scientific, Singapore 2005) 41
11. L. Lanz, B. Vacchini, and O. Melsheimer: J. Phys. A, Math. Theor. **40**(12), 3123–3140 (2007) 41

12. G. Alber, T. Beth, M. Horodecki, P. Horodecki, R. Horodecki, M. Roetteler, H. Weinfurter, R. Werner, and A. Zeilinger: *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments* (Springer, Berlin 2001) 42
13. E.B. Davies: *Quantum Theory of Open Systems* (Academic Press, London 1976) 47
14. P. Busch, M. Grabowski, and P. Lahti: *Operational Quantum Physics* (Springer-Verlag, Berlin 1995) 47
15. A. Barchielli: 'Continual measurements in quantum mechanics and quantum stochastic calculus'. In: *Open Quantum Systems. III, Volume 1882 of Lecture Notes in Mathematics*, pp. 207–292 (Springer, Berlin 2006) 54
16. R. Alicki and K. Lendi: *Quantum Dynamical Semigroups and Applications* (Springer, Berlin 1987) 58
17. R. Alicki: 'Invitation to quantum dynamical semigroups'. In: *Dynamical semigroups: Dissipation, Chaos, Quanta, Volume 597 of Lecture Notes in Physics*, pp. 239–264, edited by P. Garbaczewski and R. Olkiewicz (Springer-Verlag, Berlin 2002) 58
18. V. Gorini, A. Kossakowski, and E.C.G. Sudarshan: J. Math. Phys. **17**(5), 821–825 (1976) 61
19. G. Lindblad: Comm. Math. Phys. **48**(2), 119–130 (1976) 61
20. A.A. Budini: Phys. Rev. E **72**(5), 056106 (2005) 62
21. H.P. Breuer, J. Gemmer, M. Michel: Phys. Rev. E **73**(1), 016139 (2006) 62
22. A.A. Budini: Phys. Rev. A **74**(5), 053815 (2006) 62
23. H.-P. Breuer: Phys. Rev. A **75**(2), 022103 (2007) 62
24. H.-P. Breuer: 'Non-Markovian quantum dynamics and the method of correlated projection super-operators'. In: *Theoretical Foundations of Quantum Information Processing and Communication, Volume 787 of Lecture Notes in Physics* (Springer, Berlin) 62
25. B.-G. Englert and G. Morigi: 'Five lectures on dissipative master equations'.In: *Coherent Evolution in Noisy Environments*, Volume 611 of *Lecture Notes in Physics.*, pp. 55–106 (Springer, Berlin 2002) 62
26. B. Vacchini: J. Math. Phys. **43**(11), 5446–5458 (2002) 64, 71
27. A.S. Holevo: J. Funct. Anal. **131**(2), 255–278 (1995) 64
28. A.S. Holevo: Rep. Math. Phys. **33**(1-2), 95–110 (1993) 64
29. A.S. Holevo: Rep. Math. Phys. **32**(2), 211–216 (1993) 65, 71
30. A.Y. Artemev: Teoret. Mat. Fiz. **79**(3), 323–333 (1989) 67
31. M. Verri and V. Gorini: J. Math. Phys. **19**(9), 1803–1807 (1978) 68
32. A. Săndulescu and H. Scutaru: Ann. Phys. **173**(2), 277–317 (1987) 68
33. B. Vacchini: Phys. Rev. Lett. **84**(7), 1374–1377 (2000) 68, 73
34. R.F. O'Connell: Phys. Rev. Lett. **87**(2), 028901 (2001) 68
35. B. Vacchini: Phys. Rev. Lett. **87**(2), 028902 (2001) 68
36. A. Bassi, E. Ippoliti, and B. Vacchini: J. Phys. A **38**(37), 8017–8038 (2005) 69
37. A.S. Holevo: Izv. Math. **59**(2), 427–443 (1995) 71
38. A.S. Holevo: 'Covariant quantum dynamical semigroups: unbounded generators. In: *Irreversibility and Causality (Goslar, 1996)*, volume 504 of *L*ecture Notes in Physics, pp. 67–81 (Springer, Berlin, 1998) 71
39. W. Feller: *An Introduction to Probability Theory and its Applications. Vol. II.*, Second edition. (John Wiley & Sons Inc., New York 1971) 71, 72, 75
40. B. Vacchini: Phys. Rev. E **63**(6), 066115 (2001) 73
41. B. Vacchini: J. Math. Phys. **42**(9), 4291–4312 (2001) 73
42. B. Vacchini: Phys. Rev. E **66**(2), 027107 (2002) 73
43. F. Petruccione and B. Vacchini: Phys. Rev. E **71**(4), 046134 (2005) 74
44. K. Hornberger: Phys. Rev. Lett. **97**(6), 060601 (2006) 74
45. B. Vacchini: Phys. Rev. Lett. **95**(23), 230402 (2005) 76