Eur. Phys. J. Special Topics **159**, 135–141 (2008)

DOI: 10.1140/epjst/e2008-00702-3

THE EUROPEAN
PHYSICAL JOURNAL
SPECIAL TOPICS

Kinetic description of quantum Brownian motion

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Abstract. We stress the relevance of the two features of translational invariance and atomic nature of the gas in the quantum description of the motion of a massive test particle in a gas, corresponding to the original picture of Einstein used in the characterization of Brownian motion. The master equation describing the reduced dynamics of the test particle is of Lindblad form and complies with the requirement of covariance under translations.

1 Introduction

In recent times there has been growing interest in the study of open quantum systems [1], driven by researches on both applications and foundations of quantum theory. Indeed the community of researchers involved in the subject, in spite of the precise label attached to it, ranges from physicists and chemists, to mathematicians and probabilists. As it often happens the richness of the subject allows for different approaches and biases in the treatment of the same problem. In the present paper we come back to the quest for a deep understanding of one of the apparently simplest, but also paradigmatic situations in open quantum system theory, that is to say the motion of a quantum massive test particle in a homogeneous gas. Two guiding ideas, also present in the classical approach to Brownian motion of a suspended particle by Einstein, appear in the treatment: symmetry under translations and discrete nature of matter. Following work done in [2] we will see that these two concepts reflect themselves in the mathematical properties of the mapping describing the reduced dynamics, which has to be covariant under translations, as well as in the physical meaning of the two-point correlation function of the gas appearing in the expression of the master equation, which expresses the density fluctuations in the gas.

2 Translational invariance

2.1 Microscopic Hamiltonian

As a first step we characterize the general structure of microscopic Hamiltonians accounting for a translationally invariant reduced dynamics for the test particle. We consider a test particle subject to a translationally invariant interaction with a homogeneous bath, with a potential at most linearly depending on position, e.g., a constant gravitational field. The microscopic Hamiltonian may be written as usual

$$H = H_S + H_B + V, \tag{1}$$

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where the subscripts S and B stand for system and bath respectively, while H_S and H_B satisfy the aforementioned constraints. The interaction relies on a translationally invariant potential and is of the general form

$$V = \int d^3 \boldsymbol{x} \int d^3 \boldsymbol{y} \, A_S(\boldsymbol{x}) V(\boldsymbol{x} - \boldsymbol{y}) A_B(\boldsymbol{y}), \tag{2}$$

where $A_S(\boldsymbol{x})$ and $A_B(\boldsymbol{y})$ are self-adjoint operators of system and bath respectively, expressing the coupling between the two. The invariance under translations of the potential allows to express Eq. (2) in a very simple way in terms of the Fourier transformed quantities according to

$$V = \int d^3 \boldsymbol{Q} \, \tilde{V}(\boldsymbol{Q}) A_S(\boldsymbol{Q}) A_B^\dagger(\boldsymbol{Q}). \tag{3}$$

In order to focus on a quantum description of Brownian motion we consider a density-density coupling, so that $A_{S/B}(\boldsymbol{x}) = N_{S/B}(\boldsymbol{x})$, with $N_{S/B}(\boldsymbol{x})$ the number-density operator of system and bath respectively, whose Fourier transform $\rho_{\boldsymbol{Q}}$

$$\rho_{\boldsymbol{Q}} \equiv \int d^3 \boldsymbol{x} \, e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{x}} N_B(\boldsymbol{x}) \tag{4}$$

is also called $\boldsymbol{Q}$-component of the number-density operator [3,4]. Equation (3) thus becomes

$$V = \int d^3 \boldsymbol{Q} \, \tilde{V}(\boldsymbol{Q}) A_S(\boldsymbol{Q}) \rho_{\boldsymbol{Q}}^\dagger(\boldsymbol{Q}). \tag{5}$$

Note that an interaction of the form (5), besides being translationally invariant, commutes with the number operators N_S and N_B, so that the elementary interaction events do bring in exchanges of momentum between the test particle and the environment, but the number of particles or quanta in both systems are independently conserved, thus typically describing an interaction in terms of collisions.

2.2 Quantum linear Boltzmann equation

The case of density-density coupling given by (5), when the reservoir is a free quantum gas, has been dealt with in [5–7], and the relevant test particle correlation function turns out to be the so-called dynamic structure factor [3,4]

$$S(\boldsymbol{Q}, E) = \frac{1}{2\pi\hbar} \frac{1}{N} \int dt \, e^{\frac{i}{\hbar}Et} \langle \rho_{\boldsymbol{Q}}^\dagger \rho_{\boldsymbol{Q}}(t) \rangle, \tag{6}$$

where contrary to the usual conventions, momentum and energy are considered to be positive when transferred to the test particle, on which we are now focusing our attention, rather than on the macroscopic system. The master equation then takes the form

$$\begin{aligned} \frac{d\varrho}{dt} = & -\frac{i}{\hbar}[\mathsf{H}_0, \varrho] + \frac{2\pi}{\hbar}(2\pi\hbar)^3 n_{\text{gas}} \int d^3\boldsymbol{Q} |\tilde{V}(\boldsymbol{Q})|^2 \\ & \times \Big[e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\mathsf{X}} \sqrt{S(\boldsymbol{Q}, E(\boldsymbol{Q}, \mathsf{P}))} \varrho \sqrt{S(\boldsymbol{Q}, E(\boldsymbol{Q}, \mathsf{P}))} e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\mathsf{X}} \\ & -\frac{1}{2} \left\{ S(\boldsymbol{Q}, E(\boldsymbol{Q}, \mathsf{P})), \varrho \right\} \Big], \end{aligned} \tag{7}$$

where H_0 is the free particle Hamiltonian, X and P position and momentum operator for the test particle, n_{gas} the density of the homogeneous gas, and the dynamic structure factor appears operator-valued: in fact the energy transfer in each collision, which is given by

$$E(\boldsymbol{Q}, \boldsymbol{P}) = \frac{(\boldsymbol{P} + \boldsymbol{Q})^2}{2M} - \frac{\boldsymbol{P}^2}{2M}, \tag{8}$$

with M the mass of the test particle, is turned into an operator by replacing $\boldsymbol{P}$ with P. For the case of a free gas of particles obeying Maxwell-Boltzmann statistics the dynamic structure factor takes the explicit form

$$S(\boldsymbol{Q}, E) = \sqrt{\frac{\beta m}{2\pi}} \frac{1}{Q} e^{-\frac{\beta}{8m} \frac{\left(2mE+Q^2\right)^2}{Q^2}} \tag{9}$$

with β the inverse temperature and m the mass of the gas particles.

This result has been confirmed in recent, more general work [8,9], not relying on the Born approximation, where instead of the Fourier transform of the interaction potential the full scattering amplitude describing the collisions between test particle and gas particles appear operator-valued. In the general case the master equation describing the collisional dynamics takes the following form

$$\begin{aligned}\frac{\mathrm{d}\varrho}{\mathrm{dt}} = -\frac{i}{\hbar}\left[\mathsf{H}_0, \varrho\right] + \int d^3\boldsymbol{Q} \int_{\boldsymbol{Q}^{\perp}} d^3\boldsymbol{p} \Big[& \mathrm{e}^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} L\left(\boldsymbol{p}, \mathsf{P}; \boldsymbol{Q}\right) \varrho L^{\dagger}\left(\boldsymbol{p}, \mathsf{P}; \boldsymbol{Q}\right) \mathrm{e}^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \\ & -\frac{1}{2}\left\{ L^{\dagger}\left(\boldsymbol{p}, \mathsf{P}; \boldsymbol{Q}\right) L\left(\boldsymbol{p}, \mathsf{P}; \boldsymbol{Q}\right), \varrho \right\} \Big],\end{aligned} \tag{10}$$

where the Lindblad operators $L\left(\boldsymbol{p}, \boldsymbol{P}; \boldsymbol{Q}\right)$ are given by

$$\begin{aligned}L\left(\boldsymbol{p}, \boldsymbol{P}; \boldsymbol{Q}\right) = & \sqrt{\frac{n_{\text{gas}} m}{m_*^2 Q}} f\left(\text{rel}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}, \boldsymbol{P}_{\perp\boldsymbol{Q}}\right) - \frac{\boldsymbol{Q}}{2}, \text{rel}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}, \boldsymbol{P}_{\perp\boldsymbol{Q}}\right) + \frac{\boldsymbol{Q}}{2}\right) \\ & \times \sqrt{\mu_{\text{MB}}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}} + \frac{m}{m_*}\frac{\boldsymbol{Q}}{2} + \frac{m}{M}\boldsymbol{P}_{\|\boldsymbol{Q}}\right)},\end{aligned} \tag{11}$$

involving the elastic scattering amplitude $f\left(\boldsymbol{p}_f, \boldsymbol{p}_i\right)$ and the momentum distribution function $\mu_{\text{MB}}\left(\boldsymbol{p}\right)$ of the gas momenta given by the Maxwell-Boltzmann expression. We have further denoted relative momenta as $\text{rel}\left(\boldsymbol{p}, \boldsymbol{P}\right) \equiv \left(m_*/m\right)\boldsymbol{p} - \left(m_*/M\right)\boldsymbol{P}$, with m_* the reduced mass, while the subscripts $\|\boldsymbol{Q}$ and $\perp \boldsymbol{Q}$ indicate the component of a vector (or operator) parallel and perpendicular to the momentum transfer $\boldsymbol{Q}$, so that $\boldsymbol{P}_{\|\boldsymbol{Q}} = \left(\boldsymbol{P}\cdot\boldsymbol{Q}\right)\boldsymbol{Q}/Q^2$ and $\boldsymbol{P}_{\perp\boldsymbol{Q}} = \boldsymbol{P} - \boldsymbol{P}_{\|\boldsymbol{Q}}$ respectively. Exploiting the identity

$$\begin{aligned}\frac{m}{Q}\mu_{\text{MB}}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}} + \frac{m}{m_*}\frac{\boldsymbol{Q}}{2} + \frac{m}{M}\boldsymbol{P}_{\|\boldsymbol{Q}}\right) &= \frac{m}{Q}\mu_{\text{MB}}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}} + \left(\frac{2mE\left(\boldsymbol{Q}, \boldsymbol{P}\right) + Q^2}{Q^2}\right)\frac{\boldsymbol{Q}}{2}\right) \\ &= \mu_{\text{MB}}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}\right) S\left(\boldsymbol{Q}, \boldsymbol{P}_{\|\boldsymbol{Q}}\right),\end{aligned} \tag{12}$$

where in the last line $\mu_{\text{MB}}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}\right)$ denotes the Maxwell-Boltzmann distribution over transverse momenta, the Lindblad operators can also be written

$$L\left(\boldsymbol{p}, \boldsymbol{P}; \boldsymbol{Q}\right) = \sqrt{\frac{n_{\text{gas}}}{m_*^2}} f\left(\text{rel}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}, \boldsymbol{P}_{\perp\boldsymbol{Q}}\right) - \frac{\boldsymbol{Q}}{2}, \text{rel}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}, \boldsymbol{P}_{\perp\boldsymbol{Q}}\right) + \frac{\boldsymbol{Q}}{2}\right) \sqrt{\mu_{\text{MB}}\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}\right)} \sqrt{S\left(\boldsymbol{Q}, \boldsymbol{P}\right)},$$

thus putting again into evidence the appearance of the dynamic structure factor, whose positivity has been exploited in order to take the square root.

The relevant correlation function for the dynamics is thus given by the Fourier transform with respect to energy of the time-dependent auto-correlation function of the operator of the bath appearing in Eq. (5). The appearance of the dynamic structure factor has an important physical meaning, linking the dynamics of the test particle to the density fluctuations in the medium, as we shall see in section 3, expressing the molecular, discrete nature of matter, that is to say one of the basic insights gained by Einstein's description of Brownian motion.

The master equation (10), or (7) when considering the Born approximation, can be seen as a quantum counterpart of the classical linear Boltzmann equation, as discussed in [2,9], in that it addresses in a quantum framework the same physical situation described by the classical linear Boltzmann equation. This is also confirmed by the fact that the diagonal matrix elements in the momentum representation of the quantum linear Boltzmann equation do give back the classical linear Boltzmann equation, obviously expressed with the quantum scattering cross section.

2.3 Translation-covariant quantum dynamical semigroups

In section 2.2 we have considered a test particle interacting through collisions with a homogeneous background gas. As stressed in section 2.1 the collisions are to be described by an interaction potential only depending on the relative coordinate, so as not to spoil invariance under translations. These two requirements lead to a natural general constraint on the structure of the mapping giving the reduced dynamics. In fact homogeneity of the bath implies that the statistical operator describing its equilibrium state commutes with the momentum operator of the bath. Similarly the considered interaction V ensures that the total Hamiltonian H commutes with the momentum operator of the whole system, which we can write as $\boldsymbol{P}_S + \boldsymbol{P}_B$. Let us now consider the reduced operator of the test particle at time t obtained by taking the trace over the bath degrees of freedom of the statistical operator of the total system. Considering a factorized initial state one has

$$\mathcal{U}_t\left[\varrho_S\right] = \mathrm{Tr}_B\left(e^{-\frac{i}{\hbar}Ht}\varrho_S \otimes \varrho_B e^{+\frac{i}{\hbar}Ht}\right).$$

Exploiting further the aforementioned constraints

$$[\varrho_B, \boldsymbol{P}_B] = 0 \qquad \text{and} \qquad [H, \boldsymbol{P}_S + \boldsymbol{P}_B] = 0,$$

one immediately has, for any vector $\boldsymbol{a} \in \mathbb{R}^3$

$$\mathrm{Tr}_B\left(e^{-\frac{i}{\hbar}Ht}e^{-\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}}\varrho_S e^{+\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}} \otimes \varrho_B e^{+\frac{i}{\hbar}Ht}\right) = e^{-\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}}\,\mathrm{Tr}_B\left(e^{-\frac{i}{\hbar}Ht}\varrho_S \otimes \varrho_B e^{+\frac{i}{\hbar}Ht}\right)e^{+\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}},$$

so that a mapping $\mathcal{U}_t$ giving the reduced dynamics must obey

$$\mathcal{U}_t\left[e^{-\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}}\varrho_S e^{+\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}}\right] = e^{-\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}}\mathcal{U}_t\left[\varrho_S\right]e^{+\frac{i}{\hbar}\boldsymbol{P}_S\cdot\boldsymbol{a}}.$$

This condition is known as covariance under translations. Focusing on the Hilbert space of the massive test particle considered in the present paper, that is to say $L^2\left(\mathbb{R}^3\right)$, the condition can be stated as follows. Given the unitary representation $\mathsf{U}(\mathbf{a}) = \exp(-i\mathbf{a}\cdot\mathsf{P}/\hbar)$, $\mathbf{a} \in \mathbb{R}^3$ of the group of translations $\mathbb{R}^3$ in $L^2\left(\mathbb{R}^3\right)$, a mapping $\mathcal{L}$ acting on the statistical operators in this space is said to be translation-covariant if it commutes with the action of the unitary representation, i.e.

$$\mathcal{L}[\mathsf{U}(\mathbf{a})\varrho\mathsf{U}^\dagger(\mathbf{a})] = \mathsf{U}(\mathbf{a})\mathcal{L}[\varrho]\mathsf{U}^\dagger(\mathbf{a}), \tag{13}$$

for any statistical operator ϱ and any translation $\mathbf{a}$. The general structure of generators of quantum dynamical semigroups complying with this covariance condition has been obtained by Holevo [10,11], and it turns out that the requirement of translation covariance puts very stringent constraints on the Lindblad operators appearing in the expression of the generator. These results, while obviously fitting in the general framework set by the famous Lindblad result [12,13], go beyond it giving much more detailed information on the possible choice of operators appearing in the Lindblad form, information conveyed by the symmetry requirements and relying on a quantum generalization of the Lévy-Khintchine formula. They therefore also provide a precious starting point for phenomenological approaches exploiting relevant physical symmetries. Referring to the papers by Holevo for the related mathematical details (see also [14] for a brief résumé), the generator can be expressed as

$$\mathcal{L}[\varrho] = -\frac{i}{\hbar}\left[H\left(\mathsf{P}\right), \varrho\right] + \mathcal{L}_G[\varrho] + \mathcal{L}_P[\varrho], \tag{14}$$

with $H(\mathsf{P})$ a self-adjoint operator which is only a function of the momentum of the test particle. The so-called Gaussian part $\mathcal{L}_G$ is given by

$$\mathcal{L}_G[\varrho] = -\frac{i}{\hbar}\left[\mathsf{Y}_0 + H_{\mathrm{eff}}(\mathsf{X},\mathsf{P}), \varrho\right] + \sum_{k=1}^{r}\left[K_k\varrho K_k^\dagger - \frac{1}{2}\left\{K_k^\dagger K_k, \varrho\right\}\right], \tag{15}$$

where

$$K_k = \mathsf{Y}_k + L_k(\mathsf{P}), \qquad \mathsf{Y}_k = \sum_{i=1}^{3} a_{ki}\mathsf{X}_i, \quad H_{\text{eff}}(\mathsf{X},\mathsf{P}) = \frac{\hbar}{2i}\sum_{k=1}^{r}(\mathsf{Y}_k L_k(\mathsf{P}) - L_k^\dagger(\mathsf{P})\mathsf{Y}_k)$$

with $k = 0,\ldots,r \leq 3$ and $a_{ki} \in \mathbb{R}$, while the remaining Poisson part takes the form

$$\mathcal{L}_P[\varrho] = \int d\mu(\boldsymbol{Q}) \sum_{j=1}^{\infty} \left[e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\mathsf{X}} L_j(\boldsymbol{Q},\mathsf{P})\varrho L_j^\dagger(\boldsymbol{Q},\mathsf{P}) e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\mathsf{X}} - \frac{1}{2}\left\{ L_j^\dagger(\boldsymbol{Q},\mathsf{P}) L_j(\boldsymbol{Q},\mathsf{P}), \varrho \right\} \right], \quad (16)$$

with $d\mu(\boldsymbol{Q})$ a positive measure. The names Gaussian and Poisson arise in connection with the different contributions in the classical Lévy-Khintchine formula [15]. In the Gaussian part the Y_k are linear combinations of the three position operators of the test particle, while the generally complex functions $L_k(\mathsf{P})$ have an imaginary part accounting for friction, typically given by a linear contribution, corresponding to a friction term proportional to velocity. In the Poisson part a continuous index $\boldsymbol{Q}$ appears, together with the usual sum over a discrete index j. The expression is characterized by the appearance of the unitary operators $\exp(i\boldsymbol{Q}\cdot\mathsf{X}/\hbar)$, expressing momentum kicks, and of the functions $L_j(\boldsymbol{Q},\mathsf{P})$, operator-valued in that they depend on the momentum operators of the test particle P.

3 Fluctuation-dissipation theorem

As already stressed the two-point correlation function appearing operator-valued in the master equation is the dynamic structure factor (6), where the Fourier transform of the number-density operator $\rho_{\boldsymbol{Q}}$, as given in (4), appears. This function is directly related to the density fluctuations in the medium, as it can be seen writing it in the following way [3]:

$$S(\boldsymbol{Q},E) = \frac{1}{2\pi\hbar}\int dt \int d^3\boldsymbol{x}\; e^{\frac{i}{\hbar}(Et-\boldsymbol{Q}\cdot\boldsymbol{x})} \frac{1}{N}\int d^3\boldsymbol{y}\; \langle N_B(\boldsymbol{y}) N_B(\boldsymbol{x}+\boldsymbol{y},t)\rangle\,, \quad (17)$$

i.e., as Fourier transform with respect to energy and momentum transfer of the time dependent density correlation function. Here the connection with density fluctuations and therefore discrete nature of matter is manifest. Introducing the real correlation functions

$$\phi^-(\boldsymbol{Q},t) = \frac{i}{\hbar N}\langle[\rho_{\boldsymbol{Q}}(t),\rho_{\boldsymbol{Q}}^\dagger]\rangle \qquad \text{and} \qquad \phi^+(\boldsymbol{Q},t) = \frac{1}{\hbar N}\langle\{\rho_{\boldsymbol{Q}}(t),\rho_{\boldsymbol{Q}}^\dagger\}\rangle, \quad (18)$$

where $\{,\}$ denotes the anticommutator, the fluctuation-dissipation theorem can be formulated in terms of the dynamic structure factor as follows

$$\begin{aligned} \phi^-(\boldsymbol{Q},t) &= -\frac{2}{\hbar}\int_{-\infty}^{0} dE\; \sin(Et/\hbar)\left(1-e^{\beta E}\right) S(\boldsymbol{Q},E) \\ \phi^+(\boldsymbol{Q},t) &= -\frac{2}{\hbar}\int_{-\infty}^{0} dE\; \cos(Et/\hbar)\coth(\beta/2E)\left(1-e^{\beta E}\right) S(\boldsymbol{Q},E). \end{aligned} \quad (19)$$

We recall that contrary to the usual perspective in linear response theory we take as positive momentum and energy transferred to the particle. The dynamic structure factor can also be directly related to the dynamic response function $\chi''(\boldsymbol{Q},E)$ [4], according to

$$S(\boldsymbol{Q},E) = \frac{1}{2\pi}\left[1-\coth\left(\frac{\beta}{2}E\right)\right]\chi''(\boldsymbol{Q},E) = \frac{1}{\pi}\frac{1}{1-e^{\beta E}}\chi''(\boldsymbol{Q},E), \quad (20)$$

the relationship leading to the important fact that while the dynamic response function is an odd function of energy, the dynamic structure factor obeys the so-called detailed balance condition

$$S(\boldsymbol{Q},E) = e^{-\beta E} S(-\boldsymbol{Q},-E), \quad (21)$$

a property granting the existence of a stationary state for the master equation [7].

The significance of the appearance of the dynamic structure factor in connection to the so-called fluctuation-dissipation theorem is to be traced back to a seminal paper by van Hove [16,17]. In fact he showed that the scattering cross-section of a microscopic probe off a macroscopic sample can be written in Born approximation

$$\frac{d^2\sigma}{d\Omega_{P'}dE_{P'}}(\boldsymbol{P}) = (2\pi\hbar)^6 \left(\frac{M}{2\pi\hbar^2}\right)^2 \frac{P'}{P}|\tilde{V}(\boldsymbol{Q})|^2 S(\boldsymbol{Q},E), \tag{22}$$

where a particle of mass M changes its momentum from $\boldsymbol{P}$ to $\boldsymbol{P}' = \boldsymbol{P}+\boldsymbol{Q}$ scattering off a medium with dynamic structure factor $S(\boldsymbol{Q},E)$. This can be seen as a formulation of the fluctuation-dissipation relationship for the case of a test particle interacting through collisions with a macroscopic fluid. The energy and momentum transfer to the particle, characterized by the expression of the scattering cross-section at l.h.s. of (22) are related to the density fluctuations of the macroscopic fluid appearing through the dynamic structure factor at r.h.s. of (22). One of the basic ideas of Einstein's Brownian motion, i.e., the discrete nature of matter, once again appears in the formulation (22) of the fluctuation-dissipation relationship.

4 Friction coefficient for quantum description of Brownian motion

We now come to the master equation for the quantum description of Einstein's Brownian motion. The requirement of translational invariance has been settled in section 2, while the connection between reduced dynamics of the test particle and density fluctuations in the medium, coming about because of its discrete nature, has been taken into account in section 3. The last step to be taken is to consider the test particle much more massive than the particles making up the gas, i.e., the Brownian limit $m/M \ll 1$, which in turn implies considering both small energy and momentum transfers, similarly to the classical case [7,18]. We therefore start from (7) and consider a free gas of Maxwell-Boltzmann particles, so that taking the limiting expression of (9) when the ratio between the masses is much smaller than one leads, of necessity as can be seen from the Gaussian contribution in Holevo's result (15) but also from previous work [19–21], to a Caldeira Leggett type master equation, however without shortcomings related to the lack of preservation of positivity of the statistical operator. The master equation takes the form

$$\frac{d\varrho}{dt} = -\frac{i}{\hbar}[\mathsf{H}_0,\varrho] - \frac{i}{\hbar}\frac{\eta}{2}\sum_{i=1}^{3}[\mathsf{X}_i,\{\mathsf{P}_i,\varrho\}] - \frac{D_{pp}}{\hbar^2}\sum_{i=1}^{3}[\mathsf{X}_i,[\mathsf{X}_i,\varrho]] - \frac{D_{xx}}{\hbar^2}\sum_{i=1}^{3}[\mathsf{P}_i,[\mathsf{P}_i,\varrho]], \tag{23}$$

with

$$D_{pp} = \frac{M}{\beta}\eta \qquad \text{and} \qquad D_{xx} = \frac{\beta\hbar^2}{16M}\eta. \tag{24}$$

The friction coefficient η is uniquely determined on the basis of the microscopic information on interaction potential and correlation function of the macroscopic system, according to

$$\eta = \frac{\beta}{2M}\frac{2\pi}{\hbar}(2\pi\hbar)^3 n_{\text{gas}} \int d^3\boldsymbol{Q}\, |\tilde{V}(\boldsymbol{Q})|^2 \,\frac{Q^2}{3} S(\boldsymbol{Q},E=0), \tag{25}$$

the factor 3 being related to the space dimensions, thus proving in a specific physical case of interest the so-called standard wisdom expecting the decoherence and dissipation rate to be connected with the value at zero energy of some suitable spectral function [22]. The transition from the general master equation (7) to the approximate expression (23) has been considered in detail in [23], where the microscopic expression for the friction coefficient has been worked out in detail for the case of a constant scattering cross section. In order to point out the connection with classical Brownian motion as described by Einstein we stress that (23) is a quantum version of the classical Kramer's equation. This can be easily seen considering the usual correspondence rules between classical and quantum mechanics, sending position to multiplication by the variable and momentum to derivation, or also considering the expression of (23) for the Wigner function, which reads

$$\frac{\partial}{\partial t} W(\boldsymbol{X}, \boldsymbol{P}) = -\frac{\boldsymbol{P}}{M} \cdot \nabla_{\boldsymbol{X}} W(\boldsymbol{X}, \boldsymbol{P}) + \eta \nabla_{\boldsymbol{P}} \cdot (\boldsymbol{P} W(\boldsymbol{X}, \boldsymbol{P})) + D_{pp} \Delta_{\boldsymbol{P}} W(\boldsymbol{X}, \boldsymbol{P}) + D_{xx} \Delta_{\boldsymbol{X}} W(\boldsymbol{X}, \boldsymbol{P}),$$

and in the strong friction limit leads to the classical Smoluchowski equation with a small quantum correction [24].

The work was partially supported by the Italian MIUR under PRIN05 (BV).

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