

# Non-Markov Processes in Quantum Theory



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## 1 Introduction

Quantum theory was born as a new mechanics, capable of providing the correct quantitative assessment of phenomena which could not find their explanation within the usual framework of classical mechanics. About a century after its introduction, many different facets and complementary presentations of the theory have been worked out. It has been put into evidence in particular that quantum theory indeed provides a new probabilistic framework for the prediction of outcomes of statistical experiments. It is therefore not only a “quantum” version of classical mechanics, it is indeed a “quantum” version of classical probability theory, containing into itself an often non trivial classical limit [1–3]. One of the most intriguing and delicate aspects of quantum theory is its irreducibly probabilistic structure, conflicting with the deterministic description we are accustomed to, as well as our everyday experience of the realization of definite events. From a classical viewpoint a probabilistic analysis is only necessary if not all degrees of freedom are under control or can be taken into account in detail. Not so for quantum theory. This state of affairs has led among others to the so-called “measurement problem”. It refers to the difficulty in reconciling the classical description for macroscopic objects and the laws of quantum theory, predicting a statistical distribution rather than definite events [4]. On turn, this problem has led to consider alternatives to quantum theory, complying with its successes but leading to a different behavior for the prediction of events, effectively suppressing superposition of macroscopic objects. Among these theories one of the most renowned classes is given by collapse models, also known as dynamical reduction models [5, 6], arisen from the seminal paper [7]. Their distinctive trait is a stochastic non-linear modification of the Schrödinger equation, which on top of the standard evolution allows for the introduction of a collapse or localization mechanism. This mechanism, once accepted, avoids the measurement problem. Importantly, this mechanism has to be

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V. Allori et al. (eds.), *Do Wave Functions Jump?*, Fundamental Theories of Physics 198,  
[https://doi.org/10.1007/978-3-030-46777-7\\_24](https://doi.org/10.1007/978-3-030-46777-7_24)

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implemented at the level of the wavefunction, so as to allow for the suppression of superpositions. Nevertheless, at the level of experimental observations, it usually cannot be distinguished from other effects leading to a vanishing contribution of coherences.

The theory of open quantum system is focused on the description of the reduced dynamics of a system interacting with other degrees of freedom, typically called environment, which are not described in detail [9, 10]. The environment therefore brings in an additional level of randomness in the dynamics, on top of the unavoidable statistical aspect brought in by quantum theory. In this framework, the suppression of superposition states in a given basis is indeed predicted for a class of models known as decoherence models [8]. It thus appears that such models, bringing in another element of probabilistic description, typically provide the same average effect as dynamical reduction models, aimed at overcoming the inherent statistical structure of any quantum dynamics. In this respect, the two fields of dynamical reduction model and open quantum system share some underlying mathematical structure. We will briefly address recent advancements in open quantum system having this perspective in mind. An important caveat to be mentioned is the fact that decoherence models do not provide a solution of the measurement problem in the sense addressed by collapse models: the suppression of macroscopic superpositions only takes place in the average and a whole statistical distribution of outcomes is predicted [11].

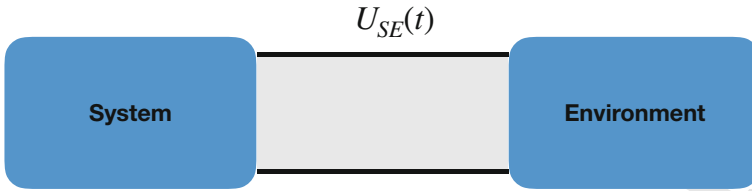
The contribution is organized as follows. In Sect. 2 we briefly outline the open quantum system viewpoint and address the term quantum process as used in the physical literature. The description of decoherence effects and their relationship to specific collapse models is worked out in Sect. 3. Finally Sect. 4 is devoted to introduce the notion of non-Markovian dynamics for an open system, and its influence on the elaboration of dynamical reduction models.

## 2 Open Systems and Quantum Processes

For the case in which a quantum system is not isolated from other quantum systems, the latter should be taken into account in the description of its dynamics. If the system and the other degrees of freedom, collectively named environment, do not share correlations at the initial time, one can describe the evolution of the system alone by introducing a collection of completely positive trace preserving maps  $\{\Phi(t)\}_{t \in \mathbb{R}_+}$ . They determine the statistics of any local observation once the initial state of the system  $\rho_S(0)$  has been specified according to the formula

$$\langle A_S \rangle_t = \text{Tr}\{A_S \Phi(t)[\rho_S(0)]\},$$

where  $A_S$  denotes a system observable. The collection of maps  $\{\Phi(t)\}_{t \in \mathbb{R}_+}$  describes what is usually called a quantum process. The term process is here used in a loose sense, in analogy with the classical situation. It hints at the presence of an irreducible randomness, here corresponding to the environmental degrees of freedom



**Fig. 1** Illustration of an open system interacting with an environment via a unitary coupling  $U_{SE}(t)$

not accessible or described in detail, but affecting the system dynamics due to a unitary coupling with the environment  $U_{SE}(t)$  as drawn in Fig. 1. If system and environment interaction can be neglected, and only in this case,  $\Phi(t)$  is a unitary transformation, implying in particular a group composition law. In all other cases irreversibility is lost, and the general mathematical structure of this collection of maps is not known. Some partial results are however available. A most famous and relevant class of reduced dynamics is obtained if we ask  $\Phi(t)$  to obey a semigroup composition law forward in time. For this case we have  $\Phi(t) = \exp(t\mathcal{L})$ , with  $\mathcal{L}$  in Lindblad form [9], that is

$$\mathcal{L}[\rho_S(t)] = -\frac{i}{\hbar}[H, \rho_S(t)] + \sum_k \lambda_k \left[ A_k \rho_S(t) A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, \rho_S(t)\} \right],$$

where  $\{A_k\}$  and  $H$  denote system operators, with  $H$  an effective self-adjoint Hamiltonian. A dynamics of this kind has always been called Markovian, since it arose as quantum counterpart of classical Markovian semigroups. The implicit idea is that the stochasticity in the dynamics arising due to interaction with the environment does not lead to effects that can be termed memory, making reference to previous history or states of the system. This feature is immediately lost even only considering dynamics which can be obtained as random mixture of unitary evolutions, so-called random unitary dynamics [12–15]. The latter might arise also as a consequence of classical environment noise and can be experimentally engineered [16, 17]. The operators  $\{A_k\}$  describe microscopic interaction events, e.g. random localization or momentum transfer events for the case of decoherence as discussed in Sect. 3.

### 3 Events and Decoherence

Dynamical reduction models and open quantum system theory share a common root in the treatment of measurement in quantum mechanics. The description of measurement deals with a description of the outcomes of statistical experiments in which the interaction with the measurement apparatus is taken into account. Indeed, the first

91 seminal contributions to open quantum systems were intimately connected with the  
 92 description of measurement processes and its relevance for the foundations of quan-  
 93 tum mechanics [18–20]. They put into evidence the relevance of the mathematical  
 94 notion of complete positivity. Not by chance the original GRW paper, which intro-  
 95 duced the first collapse model, was built upon work aimed at the quantum description  
 96 of continuous measurement in time [21, 22], and started the treatment from a master  
 97 equation describing decoherence in position [23].

98 To better work out this connection, let us consider in more detail how a collapse  
 99 model can describe in the average a decoherence effect and how a microscopic  
 100 description of decoherence can be related to a notion of event. In this spirit we  
 101 briefly recall the formulation of the GRW model in the formulation via stochastic  
 102 differential equations [5, 24]

$$103 \quad d|\psi(t)\rangle = -\frac{i}{\hbar} \hat{H}_0 |\psi(t)\rangle dt + \int_{\mathbb{R}} dy \left( \frac{L(y, \hat{x})}{\|L(y, \hat{x})|\psi(t)\rangle\|} - \mathbb{1} \right) |\psi(t)\rangle dN(y, t). \quad (1)$$

104 Here  $\psi(t)$  is the system's wavefunction,  $\hat{H}_0$  denotes the Hamiltonian appearing in  
 105 the standard Schrödinger equation and the stochastic modification is determined  
 106 by the collection of operators  $\{L(y, \hat{x})\}_{y \in \mathbb{R}}$ , with  $\hat{x}$  the standard position operator,  
 107 and the family of classical stochastic processes  $\{N(y, t)\}_{y \in \mathbb{R}}$ . Note in particular that  
 108 this modification is non-linear in  $\psi(t)$ . In order to obtain suppression of spatial  
 109 superposition of states, the  $L$  operators have to act as localization operators and to  
 110 recover the original GRW model must be of the form

$$111 \quad L(y, \hat{x}) = \frac{1}{\sqrt[4]{\pi r_c}} e^{-\frac{(y-\hat{x})^2}{2r_c^2}}. \quad (2)$$

112 The stochastic modification depends on the field of independent processes  
 113  $\{N(y, t)\}_{y \in \mathbb{R}}$  such that  $N(y, t)dy$  is the counting process giving the number of jumps  
 114 taking place at time  $t$  in the space interval from  $y$  to  $y + dy$ . The collection of count-  
 115 ing processes satisfies  $dN(x, t)dN(y, t) = \delta(x - y)dN(y, t)$ , with rates given by

$$116 \quad \mathbb{E}[dN(y, t)] = \lambda \|L(y, \hat{x})|\psi(t)\rangle\|^2 dt,$$

117 where  $\mathbb{E}[\cdot]$  denotes the stochastic average. The phenomenological parameters  $\lambda$  and  
 118  $r_c$  determine intensity and localization strength of the random jumps inducing a  
 119 dynamical localization in position of the system. Averaging over the realization of  
 120 the processes one obtains the state determining the statistics of observation on the  
 121 system, namely

$$122 \quad \rho(t) = \mathbb{E}[|\psi(t)\rangle\langle\psi(t)|],$$

123 which obeys the master equation

$$\frac{d}{dt}\rho(t) = -\lambda \left[ \rho(t) - \int dy L(y, \hat{x}) \rho(t) L(y, \hat{x}) \right] \quad (3)$$

predicting a reduction of the off-diagonal matrix elements in the position representation according to

$$\langle x | \rho(t) | y \rangle = \exp \left( -\lambda t \left[ 1 - \int dz L(z, x) L(z, y) \right] \right) \langle x | \rho(0) | y \rangle. \quad (4)$$

The obtained master Eq. (3) is in standard Lindblad form [9], describes decoherence in position according to Eq. (4), and in particular is characterised by translational invariance. Building on this aspect one realizes that it can be written in an explicit translationally covariant form [25–27] as follows

$$\frac{d}{dt}\rho(t) = -\lambda \left[ \rho(t) - \int dq \tilde{L}(q) e^{\frac{i}{\hbar} q \hat{x}} \rho(t) e^{-\frac{i}{\hbar} q \hat{x}} \right] \quad (5)$$

with  $\tilde{L}(q)$  Fourier transform of the function  $L^2(y, 0)$ , that is again a Gaussian weight. It thus appears that the dynamics that can be observed as a consequence of the localization mechanism, described at the level of trajectories of the wavefunction in Hilbert space by the stochastic differential equation Eq. (1), is the same that would arise as a consequence of interaction of the system with an external environment whose effect can be described in terms of localisation events as in Eq. (3) or in terms of momentum transfers described by the collection of unitaries  $\left\{ e^{\frac{i}{\hbar} q \hat{x}} \right\}_{q \in \mathbb{R}}$  as in Eq. (5). This viewpoint, connecting the open system based description of decoherence and the measurement based viewpoint of collapse models, implies in particular that the natural benchmark in the assessment of possible modifications of the quantum mechanical predictions due to a collapse mechanism is the estimate of possible decoherence effects affecting the considered dynamics. Indeed, this is one of the main difficulties in looking for experimental signatures of collapse mechanisms [6]. On the other hand awareness of this relationship has opened the way to consider variants of dynamical reduction models. In particular, it has led to overcome an important intrinsic limitation of models such as Eq. (1), which predict an infinite growth of the system energy [24, 28]. A further natural extension of dynamical reduction model arising from analogy and differences shared with open quantum system models is the inclusion of memory effects [29–33], in view of a definition of non-Markovian dynamics as discussed in Sect. 4.

## 4 Non-Markovian Processes

In mentioning some of the basic tenets and results of the theory of open quantum systems, we have put into evidence the notion of quantum process as used and understood in the physical literature. In particular, the time evolutions arising as solutions

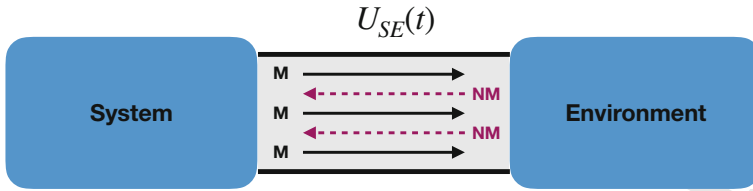
157 of master equations in Lindblad form are typically termed quantum Markovian pro-  
 158 cesses, since they provide the natural quantum counterpart of classical semigroup  
 159 evolutions, arising in connection with homogeneous in time Markovian processes.  
 160 A next natural step in this respect is considering time evolutions which can provide  
 161 a quantum realization of a non-Markovian process. Given the looser definition of  
 162 process considered in the quantum framework, as a collection of time dependent  
 163 completely positive trace preserving maps describing a continuous quantum dynam-  
 164 ics, one might consider a suitable definition of non-Markovian quantum process  
 165 within this very same framework of dynamical maps. Indeed, providing a notion of  
 166 non-Markovian quantum process in the same spirit as in the classical case, which  
 167 gives an exact definition of Markovian process in terms of conditions on the infinite  
 168 hierarchy of conditional probability densities for the process, appears to be a very  
 169 difficult task. Already from a conceptual point of view the situation does not appear  
 170 to be neatly defined, since speaking about values of an observable at a given time  
 171 calls for a measurement procedure which affects the subsequent values to be assumed  
 172 by the quantity [34]. On the contrary, focusing on the collection of completely posi-  
 173 tive trace preserving maps giving the reduced dynamics has allowed to introduce  
 174 clearcut definitions of Markovian, and in a complementary way non-Markovian,  
 175 quantum process. Actually, there have been various proposals in this direction. We  
 176 will here only focus on one of them, based on the behavior of the distinguishability  
 177 of states in time, which is in direct relationship with a notion of divisibility of the  
 178 time evolution maps. For more details and a complete treatment we refer the reader  
 179 to recent reviews [35–38].

180 The basic insight can be summarized as follows. By interacting with the envi-  
 181 ronmental degrees of freedom the system gets correlated with the environment and  
 182 possibly leads to a change in time of the reduced state of the environment itself.  
 183 As a consequence of the dynamics therefore, the capability of distinguishing two  
 184 different initial system states, by performing measurements on the system degrees of  
 185 freedom only, changes in time. Indeed, taking the partial trace necessary to define the  
 186 reduced system state, which is all that is necessary in order to provide the statistics  
 187 of measurements on the system, the whole information about correlations is no more  
 188 available. To exploit this fact one can introduce a suitable quantifier of the distin-  
 189 guishability between states, such as the trace distance, given by the trace norm of the  
 190 difference of the states

$$191 \quad D(\rho_S^1(t), \rho_S^2(t)) = \frac{1}{2} \|\rho_S^1(t) - \rho_S^2(t)\|_1 \quad (6)$$

192 and consider its behavior in time. Being a contraction under the action of completely  
 193 positive trace preserving transformations, the trace distance always diminishes with  
 194 respect to its initial value, that is

$$195 \quad D(\rho_S^1(t), \rho_S^2(t)) \leq D(\rho_S^1(0), \rho_S^2(0)).$$



**Fig. 2** Open system interacting with an environment via a unitary coupling  $U_{SE}(t)$ . Markovian effects (M) are depicted as an information flow from system to environment, while an information flow from environment to system (NM) is identified with memory effects

196 In particular for the semigroup case, considered in Sect. 2 for the case of a quantum  
 197 Markovian process, due to the composition law one has a monotonous reduction of  
 198 the distance among states with time. In such a situation the distance between states,  
 199 and therefore their distinguishability [39], gets smaller and smaller with elapsing  
 200 time. The failure of this monotonic decreasing behavior for at least a couple of possible  
 201 initial states has been taken as indication of non-Markovian dynamics in the seminal  
 202 paper [40]. Indeed, it amounts to a revival in the distinguishability between the states  
 203 that can only arise as a consequence of previously established correlations with  
 204 the environment or changes in the environmental state that affect the subsequent  
 205 reduced system dynamics. This fact is schematically drawn in Fig. 2. The validity of  
 206 this interpretation is substantiated by the inequality [41–43].

$$\begin{aligned}
 207 \quad D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(s), \rho_S^2(s)) &\leq D(\rho_{SE}^1(s), \rho_S^1(s) \otimes \rho_E^1(s)) & (7) \\
 208 \quad &+ D(\rho_{SE}^2(s), \rho_S^2(s) \otimes \rho_E^2(s)) \\
 209 \quad &+ D(\rho_E^1(s), \rho_E^2(s)),
 \end{aligned}$$

210 where it is assumed that  $t \geq s$ . The term at the lhs when positive provides a signature  
 211 of non-Markovianity, so that the positivity of the rhs is a precondition for non-  
 212 Markovianity, to be traced back to the effects mentioned above: correlations and  
 213 influence of the system on the environment. While the notions of distinguishability,  
 214 contractivity of the used distinguishability quantifier upon the action of a quantum  
 215 transformation, and connection of the distinguishability revivals to the imprint of the  
 216 system dynamics left in correlations or environment, provide the basic traits of this  
 217 approach to the description of memory effects in quantum mechanics, many more  
 218 subtle issues are involved in the definition of this framework. Importantly, there is a  
 219 stringent mathematical connection between this viewpoint and divisibility properties  
 220 of the time evolution, corresponding to the fact that the evolution over a finite time  
 221 can always be split into evolutions over shorter times, each described by a proper  
 222 quantum transformation [44–46].

223 Dynamics allowing for non-Markovian effects have also been considered in the  
 224 above-mentioned framework of a decoherence dynamics driven by random events  
 225 [47, 48], as well as in the introduction of more general dynamical reduction models  
 226 [31, 49]. While in the context of decoherence allowing for non-Markovian dynamics

227 is a way to consider more general and accurate description of the reduced dynamics,  
 228 within the framework of dynamical reduction models non-Markovian models lead to  
 229 possibly more stringent exclusion regions of the parameter values which characterise  
 230 the model.

## 231 5 Conclusions and Outlook

232 In recent times a lot of work in the field of open quantum system has been devoted to  
 233 characterization and study of non-Markovian dynamics. This research has involved  
 234 in the first instance the very definition and clarification of what can be meant as  
 235 quantum dynamics featuring memory effects. It has further addressed the possible  
 236 relevance of non-Markovian dynamics in the description of the reduced dynamics  
 237 of non isolated quantum systems as well as related fields. In this contribution we  
 238 have recalled in particular the relationship between the description of decoherence  
 239 in open quantum system and modifications of quantum mechanics such as dynamical  
 240 reduction models introduced for the sake of better grasping the so-called quantum  
 241 measurement problem. We have briefly discussed a natural physical interpretation  
 242 of non-Markovian dynamics as related to information exchange between system and  
 243 environment. We have further pointed to the use of the formalism of non-Markovian  
 244 dynamics to consider more general collapse model which might help in improving  
 245 the known bounds on the parameters characterizing the possible deviations from  
 246 standard quantum mechanics. The relevance of the classification of non-Markovian  
 247 dynamics itself as well as the role of memory effects in collapse mechanisms remain  
 248 two open questions that will surely involve future research.

249 **Acknowledgements** The author acknowledges support from the Joint Project “Quantum Infor-  
 250 mation Processing in Non-Markovian Quantum Complex Systems” funded by FRIAS, University  
 251 of Freiburg and IAR, Nagoya University, from the FFABR project of MIUR and from the Unimi  
 252 Transition Grant H2020.

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