Decoherence due to scattering events and Lévy processes

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Abstract. We point out that a unified theoretical framework for the explanation of quite different decoherence experiments can be outlined, provided the interaction causing decoherence can be described in terms of scattering events corresponding to random momentum transfers between system and environment. This unified description relies on a direct correspondence between the different possible master-equations describing such decoherence dynamics and the classical Lévy-Khintchine formula. The loss of coherence due to interaction with the environment is then described in terms of the characteristic function of the related Lévy process, accounting for a wide range of different possible behaviours.

1. Introduction

The phenomenon of decoherence (see [1, 2, 3] for a general reference), to be understood as a dynamical suppression of typically quantum interference effects due to interaction between the observed quantum system and some external environment, has turned out to be of great interest in the last few years. In fact apart from disclosing useful scenarios for a better description of the border between quantum and classical mechanics, decoherence is also directly relevant for theoretical and experimental studies on realistic implementations of quantum computing. In a wider framework it provides reference benchmarks for the validity and possible observation of theories extending or replacing quantum mechanics in the attempt to reconcile classical and quantum world.

Despite the phenomenon of decoherence in its full generality is obviously ubiquitous, since the perfect shielding of a quantum system from the rest of the universe is impossible, it seems not realistic to assume that a universal theory of decoherence can be conceived. In fact for a truly quantitative description of decoherence phenomena it appears unavoidable to have a grasp on both properties of the environment and detailed interaction mechanisms between system and environment. Nevertheless, general classes of physical situations can be considered where a common theoretical framework still applies to the details of the environment or of the relevant interaction fixing the freedom left in the theoretical description. One such case has been considered in [4], where situations in which decoherence is to be described in terms of independent random momentum transfers are dealt with in a unified way pointing to a general connection with the possible expressions of the characteristic function of a classical Lévy process for the actual estimate of the loss of visibility.

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2. Translation-covariant master-equations

We now first show that a general characterization of the structure of mappings describing a dynamics consisting of independent scattering events, each one to be expressed as a random momentum transfer between system and environment, can be given. The result relies on the fact that such mappings have to be covariant under translations, provided the environment is homogeneous, so that the random momentum transfers do not depend on the location of the scattering event, and builds on an interesting connection with classical Lévy processes. The covariance of the mapping corresponds to the requirement that its action has to commute with the action of the unitary representation of translations. Let $\mathcal{L}$ be the mapping describing the dynamics in Schrödinger picture, thus acting on the statistical operator $\hat{\rho}$ associated to the centre of mass degrees of freedom of the massive particle, whose interaction with the environment we are going to investigate. In order to be covariant $\mathcal{L}$ has to satisfy the requirement

$$\mathcal{L}\left[\mathrm{e}^{-\frac{i}{\hbar}\boldsymbol{A}\cdot\widehat{\mathsf{P}}}\hat{\rho}\mathrm{e}^{\frac{i}{\hbar}\boldsymbol{A}\cdot\widehat{\mathsf{P}}}\right] = \mathrm{e}^{-\frac{i}{\hbar}\boldsymbol{A}\cdot\widehat{\mathsf{P}}}\mathcal{L}\left[\hat{\rho}\right]\mathrm{e}^{\frac{i}{\hbar}\boldsymbol{A}\cdot\widehat{\mathsf{P}}} \qquad \forall \boldsymbol{A}\in\mathbb{R}^3, \tag{1}$$

where $\widehat{\mathsf{P}}$ denotes the momentum operator of the massive particle.

The general structure of a Markovian master-equation satisfying this covariance requirement has been worked out by Holevo (see [5] for a general reference and [6, 7, 8, 9] for applications in the physical literature), leading to an expression complying with the well-known Lindblad structure, but at the same time providing a more detailed characterization. It is given by the following expression

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}[\widehat{\mathsf{H}}(\widehat{\mathsf{P}}),\hat{\rho}] + \mathcal{L}_G[\hat{\rho}] + \mathcal{L}_P[\hat{\rho}], \tag{2}$$

where the symbols G and P denote a Gaussian and a Poisson component, the names arising from the connection with the classical Lévy-Khintchine formula. Restricting to a situation in which we neglect dissipative effects and therefore the dynamics of the momentum operator, apart from its appearance in the free kinetic term the two contributions can be written

$$\mathcal{L}_G[\hat{\rho}] = -\frac{i}{\hbar}\sum_{i=1}^{3}\mathbf{b}_i[\widehat{\mathsf{X}}_i,\hat{\rho}] - \sum_{i,j=1}^{3}\frac{1}{2}\mathbf{D}_{ij}[\widehat{\mathsf{X}}_{\mathsf{i}},[\widehat{\mathsf{X}}_j,\hat{\rho}]] \tag{3}$$

$$\mathcal{L}_P[\hat{\rho}] = \int d\boldsymbol{Q}|\lambda(\boldsymbol{Q})|^2\left[e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\widehat{\mathsf{X}}}\hat{\rho}e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\widehat{\mathsf{X}}} - \hat{\rho} - \frac{i}{\hbar}\frac{[\boldsymbol{Q}\cdot\widehat{\mathsf{X}},\hat{\rho}]}{1+Q^2}\right] \tag{4}$$

where $\mathbf{b}\in\mathbb{R}, \mathbf{D}\geq 0$, $\widehat{\mathsf{X}}$ denotes the position operator of the massive particle, and the integration measure satisfies the Lévy condition

$$\int d\boldsymbol{Q}|\lambda(\boldsymbol{Q})|^2\frac{Q^2}{1+Q^2} < \infty. \tag{5}$$

It is very convenient to write the contributions given by Eq. (3) and Eq. (4) in the position representation, leading to the simple expression

$$\langle\boldsymbol{X}|\mathcal{L}_G[\hat{\rho}] + \mathcal{L}_P[\hat{\rho}]|\boldsymbol{Y}\rangle = -\Psi\left(\boldsymbol{X}-\boldsymbol{Y}\right)\langle\boldsymbol{X}|\hat{\rho}|\boldsymbol{Y}\rangle, \tag{6}$$

where according to Eq. (3) and Eq. (4) we have introduced the function

$$\begin{aligned}\Psi\left(\boldsymbol{X}-\boldsymbol{Y}\right) &= \frac{i}{\hbar}\mathbf{b}\cdot(\boldsymbol{X}-\boldsymbol{Y}) + \frac{1}{2}(\boldsymbol{X}-\boldsymbol{Y})^T\cdot\mathbf{D}\cdot(\boldsymbol{X}-\boldsymbol{Y}) \\ &\quad - \int d\boldsymbol{Q}|\lambda(\boldsymbol{Q})|^2\left[e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot(\boldsymbol{X}-\boldsymbol{Y})} - 1 - \frac{i}{\hbar}\frac{\boldsymbol{Q}\cdot(\boldsymbol{X}-\boldsymbol{Y})}{1+Q^2}\right],\end{aligned} \tag{7}$$

only depending on the difference $(\boldsymbol{X}-\boldsymbol{Y})$ due to translational invariance. The action of the contributions given by Eq. (3) and Eq. (4) in the position representation is therefore very simple, it only amounts to multiplying the matrix elements of the statistical operator by a function of the particular form (7), whose general properties as we shall see naturally account for a description of decoherence.

3. Connection with Lévy processes and description of decoherence

The master-equation corresponding to Eq. (6) can be straightforwardly solved in the position representation, leading to the simple result

$$\langle \boldsymbol{X}|\hat{\rho}_t|\boldsymbol{Y}\rangle \;=\; \mathrm{e}^{-t\Psi(\boldsymbol{X}-\boldsymbol{Y})}\langle \boldsymbol{X}|\hat{\rho}_0|\boldsymbol{Y}\rangle. \tag{8}$$

A key point is now the observation that Eq. (7) actually gives the general expression of the characteristic exponent of the characteristic function of a Lévy process, corresponding to the celebrated Lévy-Khintchine formula [10]. As a consequence, the function

$$\Phi\left(t,\boldsymbol{X}-\boldsymbol{Y}\right) \;=\; \mathrm{e}^{-t\Psi(\boldsymbol{X}-\boldsymbol{Y})} \tag{9}$$

gives the general possible expressions for the characteristic function of a classical Lévy process. Different processes, e.g. Gaussian, Poisson, compound Poisson or Lévy stable processes arise corresponding to the different possible values of $\mathbf{b},\mathbf{D}$ and of the positive weight $|\lambda(\boldsymbol{Q})|^2$ in the measure. The function $\Phi\left(t,\boldsymbol{X}-\boldsymbol{Y}\right)$ is thus a characteristic function, so that it has the following interesting properties, explaining why Eq. (8) generally gives a well defined master-equation describing loss of coherence in the position representation:

- $\Phi\left(t,0\right)=1$
- $|\Phi\left(t,\boldsymbol{X}-\boldsymbol{Y}\right)|\leqslant 1$
- $\Phi\left(t,\boldsymbol{X}-\boldsymbol{Y}\right)$ is positive definite
- $\Phi\left(t,\boldsymbol{X}-\boldsymbol{Y}\right)\longrightarrow 0$ for $t\to\infty$
- $\Phi\left(t,\boldsymbol{X}-\boldsymbol{Y}\right)\longrightarrow 0$ for $(\boldsymbol{X}-\boldsymbol{Y})\to\infty$, provided there exists a probability density.

These properties typical of characteristic functions[11] automatically entail that the diagonal matrix elements in the position representation are not affected with elapsing time, thus preserving normalization of the statistical operator, while the off-diagonal matrix elements are generally suppressed as expected due to decoherence. The property of being positive definite ensures preservation of positivity of the statistical operator. Furthermore for a fixed spatial distance $(\boldsymbol{X}-\boldsymbol{Y})$ the off-diagonal matrix elements in the position representation are fully suppressed for long enough interacting times, while for a fixed interaction time these off-diagonal matrix elements only go to zero if the associated process admits a proper probability density, which is not the case e.g. for a Poisson process. Depending on the particular process describing the random momentum transfers in each scattering event different characteristic functions appear, corresponding to different behaviours in the suppression of the off-diagonal matrix elements for large spatial separations. The function $|\Phi\left(t,\boldsymbol{X}-\boldsymbol{Y}\right)|$, which as we shall see is directly related to the loss of visibility in interferometric experiments testing decoherence, for a fixed interaction time t, or equivalently for a fixed time of flight through the interferometer where the system is exposed to some environment, might monotonically decrease to zero for growing values of $(\boldsymbol{X}-\boldsymbol{Y})$, or also oscillate and reach asymptotically a finite value corresponding to a residual coherence. These quite different behaviours, corresponding to a more or less effective decoherence effect, are all written in the possible expressions of the characteristic function Φ, which directly determines the loss of coherence; a particular case has been dealt with in detail in [12], where a comparison between decoherence and the dynamical reduction model envisaged by Ghirardi, Rimini and Weber is considered, also estimating the relevant order of magnitudes of the different effects.

In stating the master-equation (6) we have neglected the free evolution, thus obtaining the very simple solution (8). This might be a reasonable approximation in some cases, depending on the type of experimental setup used for the study of decoherence. In general however for a truly quantitative treatment the free evolution cannot be neglected, in that it modifies the expression for the visibility, e.g. making more or less effective scattering events corresponding to random momentum transfers taking place in different positions along an interferometer [13, 14]. To this end it is useful to consider the exact solution of the master-equation including the free evolution term

$$\langle \boldsymbol{X}|\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}|\boldsymbol{Y}\rangle = \left[\frac{i\hbar}{2M}\left(\Delta_{\boldsymbol{X}} - \Delta_{\boldsymbol{Y}}\right) - \Psi\left(\boldsymbol{X} - \boldsymbol{Y}\right)\right]\langle \boldsymbol{X}|\hat{\rho}|\boldsymbol{Y}\rangle, \tag{10}$$

which can be expressed as follows

$$\langle \boldsymbol{X}|\hat{\rho}_t|\boldsymbol{Y}\rangle = \int \frac{\mathrm{d}\boldsymbol{K}\mathrm{d}\boldsymbol{S}}{(2\pi\hbar)^3}\,\mathrm{e}^{\frac{i}{\hbar}\boldsymbol{K}\cdot\boldsymbol{S}}\mathrm{e}^{-\int_0^t \mathrm{d}t'\,\Psi\left(\boldsymbol{X}-\boldsymbol{Y}+\boldsymbol{K}\frac{t-t'}{M}\right)}\langle \boldsymbol{X}-\boldsymbol{S}|\hat{\rho}_t^S|\boldsymbol{Y}-\boldsymbol{S}\rangle, \tag{11}$$

where M is the mass of the test particle and $\hat{\rho}_t^S$ denotes the solution of the free Schrödinger equation up to time t. The general solution of Eq. (10) thus essentially takes the form of a convolution of the free solution with a smearing function given by the Fourier transform of the characteristic function of the associated Lévy process with a characteristic exponent averaged over the interaction time. Considering in particular the diagonal matrix elements of (11) corresponding to the spatial probability density and introducing the auxiliary function

$$\tilde{\Psi}\left(\boldsymbol{S}\right) = \int \frac{\mathrm{d}\boldsymbol{K}}{(2\pi\hbar)^3}\mathrm{e}^{\frac{i}{\hbar}\boldsymbol{K}\cdot\boldsymbol{S}}\mathrm{e}^{-\int_0^t \mathrm{d}t'\,\Psi\left(\boldsymbol{K}\frac{t-t'}{M}\right)} \tag{12}$$

the position probability density including decoherence effects at time t, given by $\rho_t\left(\boldsymbol{X}\right) \equiv \langle \boldsymbol{X}|\hat{\rho}_t|\boldsymbol{X}\rangle$, can be exactly expressed in terms of a convolution of the free probability density $\rho_t^S\left(\boldsymbol{X}\right)$ according to

$$\rho_t\left(\boldsymbol{X}\right) = \int \mathrm{d}\boldsymbol{S}\,\tilde{\Psi}\left(\boldsymbol{S}\right)\rho_t^S\left(\boldsymbol{X}-\boldsymbol{S}\right). \tag{13}$$

Starting from Eq. (11) and relying on the Fraunhofer approximation one can obtain an expression for the visibility of the fringes expected from a two slit configuration, coming to

$$\mathcal{V} = \mathcal{V}_0 \exp\left[-\int_0^t \mathrm{d}t'\,\Re\Psi\left(\boldsymbol{d}\frac{t'-t}{t}\right)\right], \tag{14}$$

where $\mathcal{V}_0$ is a reference value of the visibility in the absence of decoherence and $\boldsymbol{d}$ is the distance between the slits. The visibility is therefore expressed in terms of the real part of the characteristic exponent Ψ averaged over interaction time. This result differs from (8) by taking into account effects due to free propagation, coming into play because of the geometry of the experimental setup.

4. Experimental realizations

After this most brief exposition of the theoretical background, we want to mention experimental situations in which such formulae have shown to be of use. The recent experiments in which a quantitative study of decoherence has been feasible typically rely on an interferometer which can be well shielded from the environment, thus permitting a clearcut observation of typical quantum effects such as interference fringes, but which also allows for a controlled, gradual inclusion of environmental interactions, not immediately and automatically spoiling all quantum features of the experiment [15, 16, 17, 18, 19, 20]. In such experiments massive particles such as sodium atoms or fullerene molecules experience random momentum transfers either because of collisions with a background gas, or because of thermal emission of electromagnetic radiation, or by absorbing and spontaneously emitting photons as a consequence of interaction with an external laser. A common feature of these apparently utterly different experimental situations is the fact that the environmental interactions consist in very few scattering events causing a certain momentum transfer between system and environment, each of these events however does not correspond to the exchange of a deterministically fixed amount of momentum. The momentum transfer is rather a random variable described by a probability distribution depending on the interaction mechanism. In the framework outlined above this situation corresponds to a compound Poisson process, that is to say a particular realization of Eq. (4) where the weight $|\lambda(\boldsymbol{Q})|^2$ in the measure takes the role of the probability density of having a definite momentum transfer, let us call it $\mathcal{P}(\boldsymbol{Q})$, the last term of Eq. (4) not contributing, as well as terms of the form of Eq. (3). The master-equation is thus given by the expression

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \Lambda \int d\boldsymbol{Q}\mathcal{P}(\boldsymbol{Q}) \left[e^{\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\mathsf{X}}}\hat{\rho}e^{-\frac{i}{\hbar}\boldsymbol{Q}\cdot\hat{\mathsf{X}}} - \hat{\rho} \right], \tag{15}$$

where Λ is a scattering rate fixed by the experimental conditions (e.g. density of background gas or laser intensity), together with $\mathcal{P}(\boldsymbol{Q})$ which is determined by the microscopic interaction mechanism, described either on the base of a microscopic theory or relying on a suitable Ansatz. Free solution and visibility as given by (8) and (14) now take the form

$$\langle \boldsymbol{X}|\hat{\rho}_t|\boldsymbol{Y}\rangle = \mathrm{e}^{-\Lambda[1-\Phi_{\mathcal{P}}(\boldsymbol{X}-\boldsymbol{Y})]t}\langle \boldsymbol{X}|\hat{\rho}_0|\boldsymbol{Y}\rangle \tag{16}$$

and

$$\mathcal{V} = \mathcal{V}_0 \exp\left\{-\Lambda\left[1-\frac{1}{t}\int_0^t \mathrm{d}t'\,\Phi_{\mathcal{P}}\left(\boldsymbol{d}\frac{t'-t}{t}\right)\right]t\right\} \tag{17}$$

respectively. The quantity $\Phi_{\mathcal{P}}$ denotes the characteristic function of the probability density $\mathcal{P}$, i.e. its Fourier transform, in term of which the characteristic function of the associated compound Poisson is expressed. In fact if for a Poisson process characterized by a rate Λ and a fixed momentum transfer $\boldsymbol{P}_0$ the characteristic function takes the usual form

$$\Phi(t,\boldsymbol{Z}) = \exp\left\{-\Lambda\left[1-\mathrm{e}^{\frac{i}{\hbar}\boldsymbol{P}_0\cdot\boldsymbol{Z}}\right]t\right\}, \tag{18}$$

for a compound Poisson process composed according to a probability distribution $\mathcal{P}(\boldsymbol{Q})$ the characteristic function correspondingly takes the form [10]

$$\Phi_{\mathcal{P}}(t,\boldsymbol{Z}) = \exp\left\{-\Lambda\left[1-\int \mathrm{d}\boldsymbol{P}\,\mathrm{e}^{\frac{i}{\hbar}\boldsymbol{P}\cdot\boldsymbol{Z}}\mathcal{P}(\boldsymbol{P})\right]t\right\} = \exp\left\{-\Lambda\left[1-\Phi_{\mathcal{P}}(\boldsymbol{Z})\right]t\right\}. \tag{19}$$

One is now facing a much more general situation than the usual "Gaussian common lore" [21] predicting a loss of coherence given by a Gaussian function with an argument linearly depending on time and quadratically depending on the path separation, even though as also shown experimentally and clarified theoretically the two different regimes can go from the one into the other [17, 4]. Depending on $\mathcal{P}(\boldsymbol{Q})$ and above all on the typical relevant values of momentum transfers in the experiment, the quantity $\Phi_{\mathcal{P}}$ takes on quite different expressions and has to be evaluated over a range of values where it might actually vary or also always be very close to either one or zero. The probability density $\mathcal{P}(\boldsymbol{Q})$ depends on its turn on the detailed interaction mechanism, so that for example in the case of a collisional interaction it is determined from the scattering cross-section between test particle and particles of the background gas.

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