Dynamical reduction models and the energy conservation principle

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Abstract. A common feature of dynamical reduction models is the violation of energy conservation principle which usually shows up as a constant increase in time of the energy of isolated systems. Anyway for typical values of the parameters of the models, such a violation is usually so weak that cannot be detected with present-day technology. . Despite the reduction mechanism itself seems responsible for this behaviour, we show that this is not a intrinsic property of dynamical reduction models: we exhibit a collapse model such that the energy of isolated systems does not diverge for large times but reaches an asymptotic finite value. This result could be interesting in understanding how to work out relativistic extensions of dynamical reduction models.

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INTRODUCTION

Dynamical reduction models (for a review see Ref. [1]) are considered one of the most promising attempts to overcome the so-called macro-objectification or measurement problem of quantum mechanics. The strategy they follow is to suitably modify the Schrödinger equation in such a way to account for the behaviour of both microscopic and macroscopic systems: the evolution of the wavefunction is described in terms of the following nonlinear but norm-preserving stochastic differential equations:

$$d\psi_t = \left[-\frac{i}{\hbar} H\, dt + \sqrt{\lambda}\,(A - r_t)\, dW_t - \frac{\lambda}{2}\left(A^\dagger A - 2 r_t A + r_t^2\right) dt \right] \psi_t, \qquad (1)$$

with:

$$r_t \;=\; \frac{1}{2}\,\langle \psi_t | (A + A^\dagger) | \psi_t \rangle .$$

The operator H, as we will see, is related to the standard quantum Hamiltonian of the system, while A is the *reduction operator*, i.e., the operator on whose eigenmanifolds one wants to reduce the statevector, as a consequence of the collapse mechanism; λ is a positive constant controlling the speed of the reduction process and W_t is a standard Wiener process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Eq. (1) can be possibly generalized by adding any number of operators A_i, each of which is coupled to a Wiener process W_i, or by considering complex Wiener process.

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All most important dynamical reduction models analysed in the literature, which aim at localizing wavefunction in space, exhibit the typical feature of violating the energy conservation for isolated systems. In fact if one chooses A to be the position operator q or a function of it, like in the continuous version [3] of the original GRW model [2] (the reason being that the operator q is the most natural candidate for localizing wavefunctions in space), then it can be shown [4] that the reduction mechanism induces larger and larger fluctuations in the momentum space as one can roughly infer by using the uncertainty principle: the smaller the uncertainty in position the bigger the uncertainty in momentum. Such fluctuations, in turn, determine the growth of the energy of the system, which eventually diverges for large times.

Then, apparently, the energy nonconservation appears to be an intrinsic feature of dynamical reduction models since the stochastic process driving the reduction mechanism seems to be directly responsible for such a violation. In spite of the fact that at nonrelativistic level, for typical values of the parameters of the models, the unlimited growth of the energy takes place with such a small rate that it can not be detected by present-day technology, it is natural to wonder if such a behaviour is an *unavoidable* inherent property of dynamical reduction models or it can be avoided with a suitable choice of the reduction operators. A resolution to this problem could be important if we consider that the difficulties in finding a coherent relativistic generalization of collapse models stem precisely from the appearance of non-renormalizable divergent quantities such as the mean energy.

THE MODEL

In [18] we have proposed a new model of spontaneous wavefunction collapse such that the mean energy of isolated systems does not increase constantly in time but reaches an asymptotic finite value $\overline{E}$; as we will see, such a value is related to a temperature T which can be regarded as the temperature of the noise responsible for the collapse mechanism. The idea we have followed is to choose the localization operator A in such a way that the noise, besides localizing, contains also a friction term which absorbs the energy when it reaches too a high value.

The choice we made for the reduction operator is the following:

$$A = q + i\frac{\tau}{\hbar}p, \tag{2}$$

where q is the position operator and p the momentum operator; moreover, we have defined the operator H as follows:

$$H = H_0 + \frac{\lambda\tau}{2}\{q,p\}, \tag{3}$$

where H_0 is the standard quantum Hamiltonian.

As we can see from the Eqs (1) and (2) our model is defined in terms of the two positive constant: λ, which sets the strength of the collapse mechanism, and τ, which measures the dissipative effect of the noise. For reasons that will be clear in the subsequent analysis, we will assume such constants to be functions of the mass of the particle,

as follows:

$$\lambda = \frac{m}{m_0}\lambda_0 \qquad\qquad \tau = \frac{m_0}{m}\tau_0, \tag{4}$$

where m is the total mass of the system and m_0 is a reference mass which we choose to be equal to that of a nucleon, while λ_0 and τ_0 are fixed constants which we assume to be of order:

$$\lambda_0 \simeq 10^{-2}\,\mathrm{m}^{-2}\,\mathrm{sec}^{-1}, \tag{5}$$

$$\tau_0 \simeq 10^{-18}\,\mathrm{m}^2. \tag{6}$$

As we will see in the following sections, the above numerical values have been chosen in such a way to guarantee that the model preserves the two essential requirements necessary for a dynamical reduction model to be consistent:

1. the model must reproduce almost exactly the physical predictions of standard quantum mechanics at the microscopic level;
2. the model must reproduce classical mechanics at the macroscopic level.

THE EVOLUTION OF THE MEAN ENERGY

We begin our analysis by presenting the main feature of our model, i.e. we discuss how the mean energy of isolated systems evolves in time.

The energy operator of an isolated particle of mass m is $H_0 = p^2/2m$, and the meaningful physical quantity to compute is the time evolution of $\mathbb{E}[\langle H_0\rangle_t] \equiv \mathbb{E}[\langle\psi_t|H_0|\psi_t\rangle]$, where $\mathbb{E}$ represents the statistical average with respect to the probability measure $\mathbb{P}$. It is not difficult to show that Eq. (1) with the choice (2) for the localization operator yields the following equation for the mean energy:

$$\frac{d}{dt}\mathbb{E}[\langle H_0\rangle_t] = \frac{\lambda\hbar^2}{2m} - 4\lambda\tau\mathbb{E}[\langle H_0\rangle_t], \tag{7}$$

whose solution is:

$$\mathbb{E}[\langle H_0\rangle_t] = \left(E_0 - \frac{\hbar^2}{8m\tau}\right)e^{-4\lambda\tau t} + \frac{\hbar^2}{8m\tau}. \tag{8}$$

As we see, the mean energy of an isolated system is still not conserved, but now it does not diverge for $t \to +\infty$, on the contrary it reaches the asymptotic finite value:

$$\overline{E} \equiv \lim_{t\to+\infty}\mathbb{E}[\langle H_0\rangle_t] = \frac{\hbar^2}{8m\tau} = \frac{\hbar^2}{8m_0\tau_0}. \tag{9}$$

Moreover the time evolution is very slow, the rate of change being equal to $4\lambda\tau = 4\lambda_0\tau_0 \simeq 10^{-20}\,\mathrm{sec}^{-1}$, which means that, with very high accuracy, the mean energy remains constant and equal to the initial value for very long times; one can then say that the energy conservation principle is valid for all practical purposes and for very long times.

In order to better understand why Eqs. (1) and (2) yield to such a result, it is worthwhile considering the related equation for the statistical operator $\rho_t \equiv \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$, which is given by:

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}\left[H_0,\rho_t\right] - \frac{\lambda}{2}\left[q,[q,\rho_t]\right] - \frac{\lambda\tau^2}{2\hbar^2}\left[p,[p,\rho_t]\right] - i\frac{\lambda\tau}{\hbar}\left[q,\{p,\rho_t\}\right]. \tag{10}$$

In the above equation one can recognize the typical structure of the master-equation for the quantum description of Brownian motion, where both friction and diffusion are taken into account and positivity of the statistical operator is granted at all times. As a matter of fact if we define:

$$\beta \equiv \frac{4m\tau}{\hbar^2}, \qquad\qquad \gamma \equiv \lambda\tau \tag{11}$$

Eq. (10) becomes precisely the quantum Brownian motion master-equation[5, 6, 7]:

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}\left[H_0,\rho_t\right] - \gamma\frac{2m}{\beta\hbar^2}\left[q,[q,\rho_t]\right] - \gamma\frac{\beta}{8m}\left[p,[p,\rho_t]\right] - \frac{i}{\hbar}\gamma\left[q,\{p,\rho_t\}\right], \tag{12}$$

where γ is the friction coefficient and β the inverse temperature of the background medium; the second and third term at r.h.s. account for diffusion, with coefficients proportional to the squared thermal wavelength $\Delta x_{\rm th}^2 = \beta\hbar^2/4m$ and the squared thermal momentum spread $\Delta p_{\rm th}^2 = m/\beta$ satisfying the minimum uncertainty relation $\Delta p_{\rm th}\Delta x_{\rm th} = \hbar/2$, while the last term is due to friction and ensures that the energy of the test particle asymptotically goes to the equipartition value depending only on the temperature of the bath. Note that in the quantum description, friction, which accounts for the finite energy growth, is of necessity related to diffusion in order to preserve the Heisenberg uncertainty relation [8, 5, 6].

One might wonder what is the origin of the particular expression (2) for the reduction operator leading to Eq. (10) for the statistical operator, which has the same operator structure as the quantum Brownian motion master-equation (12), and whether more general expressions can also be considered. To clarify this point we stress that a natural requirement on dynamical reduction models is that the reduction mechanism should not break the homogeneity of space, so that as a consequence the related equations should not violate translational invariance. In particular this property should hold for the resulting master-equation for the statistical operator. Mathematically speaking this amounts to the requirement that the generator of the time-evolution $\mathscr{L}$ characterizing the operator structure of the master-equation should be translation-covariant, i.e., it should commute with the action of the unitary representation of translations $U(a) = \exp[-(i/\hbar)ap]$

$$\mathscr{L}[U(a)\rho U^\dagger(a)] = U(a)\mathscr{L}[\rho]U^\dagger(a) \tag{13}$$

for all real a. Starting from this requirement Holevo has obtained a characterization of generators of quantum-dynamical semigroups covariant under translations [9, 10, 11, 12]. His results show that in this case one still obtains a Lindblad structure as naturally expected

$$\mathscr{L}[\rho_t] = -\frac{i}{\hbar}\left[H,\rho_t\right] + \sum_i \left[L_i\rho L_i^\dagger - \frac{1}{2}\left\{L_i^\dagger L_i,\rho\right\}\right], \tag{14}$$

however the requirement of translational invariance gives important restrictions on the actual form of the operators H and L_i appearing in Eq. (14), also implicitly suggesting their possible expressions. If in particular one wants to extend a dynamical reduction model where the reduction operator is simply given by the position operator as considered for example in [4], the most natural and simple extension still preserving translational invariance is given by Eq. (2), corresponding at the master-equation level to a friction term proportional to velocity. Other possibilities would imply by necessity a friction effect proportional to higher powers of velocity or a dependence other than linear (actually given by an imaginary exponential, thus leading to a unitary operator) on the position operator.

The above considerations suggest that the stochastic process in Eq. (1) acts like a dissipative medium which, due to friction, slowly thermalizes all systems to the temperature

$$T = \frac{1}{k_B \beta} = \frac{\hbar^2}{4 m_0 \tau_0 k_B} \simeq 10^{-1}\ \mathrm{K}, \tag{15}$$

by absorbing or transferring energy to them depending on their initial state. Note that according to Eq. (15), and as we will see in the next section, a very "cold" medium is enough to guarantee the localization in space of the wavefunctions of macroscopic objects. Note also that one recovers a GRW-type equation by setting $\tau \to 0$, which corresponds to the high temperature limit $T \to +\infty$. This shows that the reason why in the original GRW reduction model [2] the energy increases and eventually diverges is that the noise acts like an infinite-temperature stochastic medium.

The analogy with quantum Brownian motion suggests that the model could be further developed by promoting W_t to a *real physical* medium with its own equations of motions, having a stochastic behaviour to which a temperature T can be associated and such that, with good accuracy, can be treated like a Wiener process. This would imply not only that the medium acts on the wavefunction by changing its state according to Eq. (1), but also that the wavefunction acts back on the medium according to equations which still have to be studied. The above suggestion opens the way to the possibility that by taking into account the energy of both the quantum system *and* the stochastic medium one can restore perfect energy conservation not only on the average but also for single realizations of the stochastic process. A similar proposal has been considered by P. Pearle [13] and by S.L. Adler [14].

DYNAMICS OF MICROSCOPIC AND MACROSCOPIC SYSTEMS

Having shown that the energy evolves in the required way, we now verify that our model reproduces both the quantum regime and the classical one, i.e. we show that at the microscopic level the physical predictions of the model are almost identical to standard quantum predictions and that, at the same time, the model reproduces with high accuracy classical mechanics at the macro-level.

Micro-systems: comparison with standard quantum mechanics

Microscopic system cannot be directly observed, and their properties can be analysed only by resorting to suitable measurement procedures as discussed in detail in Ref. [19]. All physical predictions of dynamical reduction models concerning the outcome of measurements have the form $\mathbb{E}[\langle\psi_t|S|\psi_t\rangle]$ where S is a suitable self-adjoint operator, typically a projection operator, and it is easy to show that

$$\mathbb{E}[\langle\psi_t|S|\psi_t\rangle] \equiv \mathrm{Tr}[S\rho_t],$$

where the statistical operator $\rho_t \equiv \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$ satisfies Eq. (10).

As a consequence, and as discussed in the previous section, the testable effects of the stochastic process on the wavefunction are similar to the effects induced by a quantum environment on the particle, when both friction and diffusion are taken into account. With our choice (5) and (6) for λ_0 and τ_0, the testable effects of the stochastic process are of the same order of magnitude of those induced by the interaction of the system with particles and radiation of the *intergalactic space* [15]: such effects are very small and masked by most other sources of decoherence, so that they can be tested only by resorting to sophisticated experiments [16, 17]. This implies that the physical predictions of our model are very close to standard quantum mechanical predictions. For a more detailed discussion of this issue see Ref. [19].

Macro-objects: comparison with classical mechanics

A macroscopic object is made of elementary constituents strongly interacting among each other and, according to our model, its dynamics is governed by the following stochastic differential equation, which is the straightforward generalization of Eq. (1) to a system of N particles:

$$\begin{aligned} d\,\psi_t(\{x\}) = \Bigg[& -\frac{i}{\hbar} H_{\mathrm{TOT}}\, dt + \sum_{n=1}^{N} \sqrt{\lambda_n}\,(A_n - r_{nt})\, dW_t^n \\ & - \sum_{n=1}^{N} \frac{\lambda_n}{2} \left(A_n^\dagger A_n - 2 r_{nt} A_n + r_{nt}^2 \right) dt \Bigg]\, \psi_t(\{x\}); \end{aligned} \tag{16}$$

the symbol $\{x\} \equiv x_1, x_2, \ldots, x_N$ represents the N spatial coordinates of the configuration space of the composite system; $W_t^1, W_t^2, \ldots W_t^N$ are N independent Wiener processes; $r_{nt} = \langle\psi_t|[A_n + A_n^\dagger]|\psi_t\rangle/2$ and the localization operators A_n are given by expressions (2), with q replaced by q_n, the position operator of the n-th particle, and p replaced by p_n, the corresponding momentum operator. Furthermore:

$$\lambda_n = \frac{m_n}{m_0}\,\lambda_0 \qquad \tau_n = \frac{m_0}{m_n}\,\tau_0 \tag{17}$$

and

$$H_{\text{TOT}} = H^0_{\text{TOT}} + \sum_{n=1}^{N} \frac{\lambda_n \tau_n}{2} \{q_n, p_n\}, \tag{18}$$

m_n and H_0 being, respectively, the mass of the n-th particle and the standard quantum Hamiltonian for the composite system.

As custom, we separate the motion of the center of mass from the internal motion. To this end we define:

$$Q = \frac{1}{M} \sum_{n=1}^{N} m_n q_n \qquad \left(M = \sum_{n=1}^{N} m_n \right) \tag{19}$$

to be the position operator associated to the center-of-mass coordinate R, and $\tilde{q}_n$ the position operators associated to the internal coordinates $\tilde{x}_n = x_n - R$ $(n = 1, \ldots, N)$; let also P and $\tilde{p}_n$ be the corresponding momentum operators. Then, if H^0_{TOT} can be written as the sum of a term H^0_{CM} associated to the center of mass and a term H^0_{REL} associated to the internal motion, it is easy to prove that the dynamics of the two types of degrees of freedom decouple; in particular the equation for the center of mass, the only one we consider here, becomes:

$$d\,\psi_t(R) = \left[-\frac{i}{\hbar} H_{\text{CM}}\, dt + \sqrt{\lambda_{\text{CM}}}\, (A_{\text{CM}} - r_{\text{CM,t}})\, dW_t - \frac{\lambda_{\text{CM}}}{2} \left(A^{\dagger}_{\text{CM}} A_{\text{CM}} - 2 r_{\text{CM,t}} A_{\text{CM}} + r^2_{\text{CM,t}} \right) dt \right] \psi_t(R), \tag{20}$$

with:

$$H_{\text{CM}} = H^0_{\text{CM}} + \frac{\lambda_{\text{CM}} \tau_{\text{CM}}}{2} \{Q, P\}, \tag{21}$$

$$r_{\text{CM,t}} = \frac{1}{2} \langle \psi_t | [A^{\dagger}_{\text{CM}} + A_{\text{CM}}] | \psi_t \rangle, \tag{22}$$

$$A_{\text{CM}} = Q + i \frac{\tau_{\text{CM}}}{\hbar} P, \tag{23}$$

and W_t, which is defined by:

$$W_t = \sum_{n=1}^{N} \sqrt{\frac{m_n}{M}}\, W^n_t, \tag{24}$$

is easily proven to be a standard Wiener process. The two constants λ_{CM} and τ_{CM} are defined by Eqs. (4), with m now equal to the total mass M of the composite system. According to Eq. (20) the center of mass behaves like a free quantum particle with a mass equal to the total mass M of the composite system.

Note that, in the non-Schrödinger terms of Eq. (16), the separation of the center-of-mass motion from the relative motion, which leads to the Eq. (20), is possible precisely because of the specific dependence of the parameters λ_n and τ_n on the masses m_n of the particles, as given by Eq. (17).

According to Eq. (20) the center of mass behaves like a free quantum particle with a mass equal to the total mass M of the composite system. In [18] we have shown in detail why and how the different numerical values of the mass in the microscopic and

macroscopic regime are responsible for the very different behaviour of the solutions of Eq. (1) in the two cases, thus explaining why our model reproduces both classical mechanics at macro-level and quantum mechanics at micro-level. Here we want to explain such a feature of Eq. (1) by considering the time evolution of Gaussian solutions.

The time evolution of Gaussian wavefunctions

Gaussian wavefunctions are physically interesting since they often suitably represent typical physical situations; moreover they can be treated analytically without resorting to approximations. Let us therefore study in some detail the time evolution of the following class of wavefunctions:

$$\psi_t(x) = \sqrt[4]{\frac{2a_t^{\mathrm{R}}}{\pi}} \exp\left[-a_t(x-\bar{x}_t)^2 + i\left(\bar{k}_t x + \gamma_t\right)\right], \tag{25}$$

where a_t is a complex function of time[1], while γ_t, $\bar{x}_t$ and $\bar{k}_t$ are real quantities. One can quite easily show [18] that the above parameters obey the following stochastic differential equations:

$$da_t = \left[-\frac{2i\hbar}{m}a_t^2 - 4\lambda\tau a_t + \lambda\right]dt, \tag{26}$$

$$d\bar{x}_t = \frac{\hbar}{m}\bar{k}_t\,dt + \sqrt{\lambda}\left[\frac{1}{2a_t^{\mathrm{R}}} - \tau\right]dW_t, \tag{27}$$

$$d\bar{k}_t = -2\lambda\tau\bar{k}_t\,dt - \sqrt{\lambda}\,\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}dW_t.; \tag{28}$$

the equation for γ_t is not relevant since it represents only a global phase factor.

Mean in position and momentum

>From Eqs. (27) and (28) one can see that the mean $\langle q\rangle_t$ in position and the mean $\langle p\rangle_t$ in momentum satisfy the following stochastic differential equations:

$$d\langle q\rangle_t = \frac{1}{m}\langle p\rangle_t\,dt + \sqrt{\lambda}\left[\frac{1}{2a_t^{\mathrm{R}}} - \tau\right]dW_t, \tag{29}$$

$$d\langle p\rangle_t = -2\lambda\tau\langle p\rangle_t\,dt - \sqrt{\lambda}\hbar\,\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}dW_t, \tag{30}$$

from which it follows that their average values evolve as follows:

$$m\frac{d}{dt}\mathbb{E}\left[\langle q\rangle_t\right] = \mathbb{E}\left[\langle p\rangle_t\right], \tag{31}$$

[1] The superscripts "R" and "I" denote, respectively, the real and imaginary parts of the corresponding quantities.

$$\frac{d}{dt}\mathbb{E}[\langle p\rangle_t] \;=\; -2\lambda\tau\,\mathbb{E}[\langle p\rangle_t]\,. \tag{32}$$

The first equation reproduces the classical relation between position and momentum of a particle while the second equation predicts that the momentum decays exponentially in time:

$$\mathbb{E}[\langle p\rangle_t] \;=\; \langle p\rangle_0\, e^{-2\lambda\tau t}, \tag{33}$$

with a rate

$$2\lambda\tau \;=\; 2\lambda_0\tau_0 \;\simeq\; 10^{-20}\,\mathrm{sec}^{-1}, \tag{34}$$

which is extremely slow and does not depend on the mass of the particle, as it happens for the energy. We can then say that according to our model the average momentum of the free particle is approximately conserved, as it happens for an isolated system, according to Newton's law.

Spread in position and momentum

The parameter a_t appearing in (25) gives a measure of the spread of the Gaussian wavefunction both in position and momentum, according to the following expressions:

$$\sigma_q(t) \;\equiv\; \sqrt{\langle q^2\rangle - \langle q\rangle^2} = \frac{1}{2}\sqrt{\frac{1}{a_t^{\mathrm{R}}}}, \tag{35}$$

$$\sigma_p(t) \;\equiv\; \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \hbar\sqrt{\frac{(a_t^{\mathrm{R}})^2+(a_t^{\mathrm{I}})^2}{a_t^{\mathrm{R}}}}. \tag{36}$$

Eq. (26) for a_t can be easily solved obtaining:

$$a_t = -\frac{1}{2}\left[B + iD\tanh\left(\frac{\hbar}{m}Dt + F\right)\right], \tag{37}$$

with:

$$B = -2i\frac{\lambda\tau m}{\hbar}, \qquad D = \sqrt{\frac{4\lambda^2\tau^2m^2}{\hbar^2} + i\frac{2\lambda m}{\hbar}}; \tag{38}$$

the constant F sets the initial condition a_0.

The explicit solution for the standard deviation of q then is:

$$\sigma_q(t) = \sqrt{\frac{\hbar}{\sqrt{2}m\omega}\,\frac{\cosh(s_1)+\cos(s_2)}{\sin\theta\sinh(s_1)+\cos\theta\sin(s_2)}} \tag{39}$$

where we have introduced the following two quantities:

$$s_1 = \sqrt{2}\,\omega t\cos\theta + \varphi_1 \qquad s_2 = \sqrt{2}\,\omega t\,\sin\theta + \varphi_2, \tag{40}$$

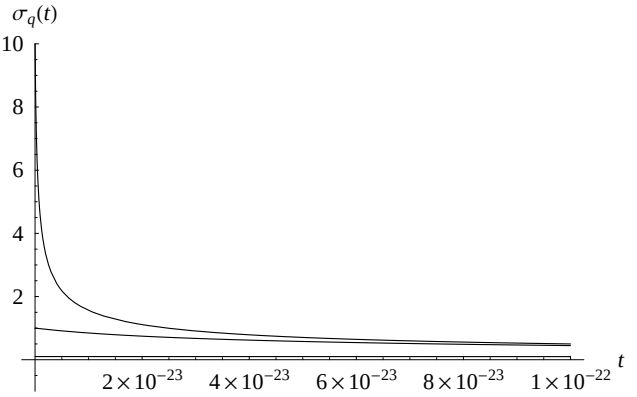

FIGURE 1. The picture shows the time evolution (t measured in seconds) of the spread in position $\sigma_q(t)$ (measured in meters) of a minimum-uncertainty Gaussian wavefunction for different initial conditions: $\sigma_q(0) = 10$ m, $\sigma_q(0) = 1$ m, $\sigma_q(0) = 0.1$ m (from the upper to the lower curve). The simulation has been made for an object of $m = 1$g.

as functions of the constants

$$\omega = 2\sqrt[4]{4\lambda_0^4\alpha_0^4 + \frac{\lambda_0^2\hbar^2}{m_0^2}} \simeq 10^{-5}\ \mathrm{sec}^{-1}, \tag{41}$$

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{\hbar}{2\lambda_0\alpha_0^2 m_0}\right] \simeq \frac{\pi}{4}, \tag{42}$$

and the two parameters φ_1 and φ_2 which are functions of the initial conditions.

According to (39), the spread in position of the Gaussian wavefunction (25) evolves deterministically in time and depends on the noise W_t only indirectly, through the constant λ. Moreover, while in the standard quantum case the spread in position of the wavefunction of a free quantum particle increases in time, diverging for $t \to \infty$, the spread according to our model reaches the asymptotic *finite* value:

$$\overline{\sigma}_q \equiv \sigma_q(\infty) = \sqrt{\frac{\hbar}{\sqrt{2}m\omega\sin\theta}} \simeq \left(10^{-15}\sqrt{\frac{\mathrm{Kg}}{m}}\right)\mathrm{m}. \tag{43}$$

For a microscopic object like a nucleon such an asymptotic spread is of order:

$$\overline{\sigma}_q \simeq 10^{-2}\ \mathrm{m} \qquad \text{for a nucleon,} \tag{44}$$

while in the case of a macroscopic object like a tennis ball it becomes extremely small:

$$\overline{\sigma}_q \simeq 10^{-14}\ \mathrm{m} \qquad \text{for a tennis ball.} \tag{45}$$

As one can see in Fig. 1, wavefunctions of macroscopic objects are rapidly localized in space; Fig. 2 instead shows how the collapse effect becomes faster for increasing values of the mass m.

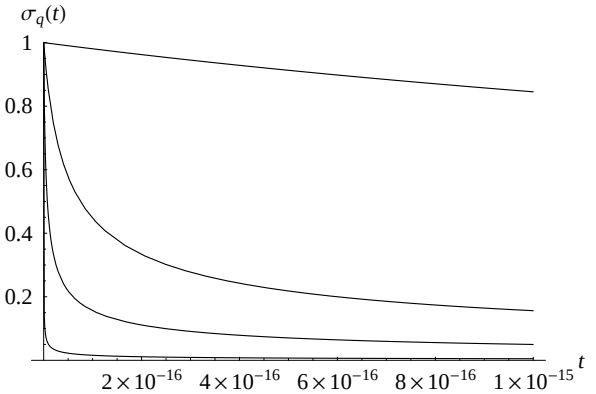

FIGURE 2. The picture shows the time evolution (t measured in seconds) of the spread in position $\sigma_q(t)$ (measured in meters) of a minimum-uncertainty Gaussian wavefunction for particles of different masses: $m = 10^{-3}$ g, $m = 10^{-5}$ g, $m = 10^{-6}$ g and $m = 10^{-8}$ g (from the lower to the upper curve). The simulation has been made assuming the initial spread $\sigma_q(0) = 1$ m.

>From Eq. (36) we can also obtain the evolution equation for $\sigma_p(t)$: as it happens for the spread in position, also the spread in momentum asymptotically reaches a finite value, which is:

$$\overline{\sigma}_p \equiv \sigma_p(\infty) = \sqrt{\frac{\hbar m\omega}{2\sqrt{2}}\,\frac{\sin^2\theta + (\cos\theta - \kappa)^2}{\sin\theta}} \simeq \left(10^{-19}\sqrt{\frac{m}{\mathrm{Kg}}}\right)\frac{\mathrm{Kg\ m}}{\mathrm{sec}} \tag{46}$$

with

$$\kappa = \frac{2\sqrt{2}\lambda_0\alpha_0}{\omega} \simeq 10^{-14}. \tag{47}$$

If we consider once more the two examples previously used, we have for the asymptotic spread of the *velocity* in the micro and macro regime the following order of magnitude:

$$\overline{\sigma}_v \simeq 10^{-6}\ \mathrm{m/sec} \qquad \text{for a nucleon,} \tag{48}$$

$$\overline{\sigma}_v \simeq 10^{-18}\ \mathrm{m/sec} \qquad \text{for a tennis ball.} \tag{49}$$

We see that in the macroscopic case wavefunctions are well localized not only in the position space, but also in the momentum space.

Fluctuations around the mean

We have seen above that any Gaussian solution converges towards a Gaussian wavefunction having the asymptotic values (43) and (46) for the spread in position and momentum: such a Gaussian solution is sometimes called a "stationary solution" of Eq. (1). Of course, the term "stationary" does not mean that such wavefunctions do not evolve

in time; as a matter of fact they always undergo a random motion both in position and momentum which never stops[2].

One can compute [18] the flactuations of both $\langle q\rangle_t$ and $\langle p\rangle_t$ for these special solutions:

$$\mathbb{E}\left[\left(\langle q\rangle_t - \mathbb{E}[\langle q\rangle_t]\right)^2\right] \simeq \frac{4\hbar^2\lambda_0}{m_0\omega^2}\frac{t}{m} \simeq 10^{-33}\left(\frac{\mathrm{Kg}}{m}\right)\left(\frac{t}{\mathrm{sec}}\right)\mathrm{m}^2 \tag{50}$$

$$\mathbb{E}\left[\left(\langle p\rangle_t - \mathbb{E}[\langle p\rangle_t]\right)^2\right] \simeq \frac{\hbar^2\lambda_0}{m_0}mt \simeq 10^{-43}\left(\frac{m}{\mathrm{Kg}}\right)\left(\frac{t}{\mathrm{sec}}\right)\frac{\mathrm{Kg}^2\mathrm{m}^2}{\mathrm{sec}^2}. \tag{51}$$

which are valid when $t \ll (\lambda_0\tau_0)^{-1} \simeq 10^{20}\mathrm{sec} \simeq 10^{12}\mathrm{years}$.

The rates of such a fluctuations are very small quantities, when m is the mass of a macro-object as for our tennis ball:

$$\mathbb{E}\left[\left(\langle q\rangle_t - \mathbb{E}[\langle q\rangle_t]\right)^2\right]/t \simeq 10^{-31}\ \mathrm{m}^2, \tag{52}$$

$$\mathbb{E}\left[\left(\langle p\rangle_t - \mathbb{E}[\langle p\rangle_t]\right)^2\right]/t \simeq 10^{-41}\ \frac{\mathrm{Kg}^2\mathrm{m}^2}{\mathrm{sec}^2}. \tag{53}$$

Accordingly, in the macroscopic regime the actual value of the two peaks in position and momentum of a stationary Gaussian solution are very close to their average values which, as we have seen, evolve in time according to Newton's laws for a free particle moving in a (very weakly) dissipative medium.

To summarize we have obtained the three following results regarding the behaviour of the center of mass of a macroscopic object:

1. A Gaussian wavefunction (indeed, as shown in [18] almost any initial wavefunction) very rapidly converges to a stationary solution, which is very well localized both in position and in momentum.
2. The fluctuations around the averages of both the mean in position and in momentum, for a stationary solution, are very weak, almost negligible.
3. The average values as shown by Eqs. (31) and (32) evolve in time according to Newton's laws.

This shows with very high accuracy that the typical wavefunction for the center of mass of a macro-system practically behave like a point moving "deterministically" in space according to the laws of classical mechanics.

CONCLUSIONS

We have shown that the indefinite increase of the mean energy of a quantum system subject to spontaneous localizations is not an intrinsic feature of dynamical reduction

[2] The term "stationary" refers only to the shape of the wavefunction: stationary solutions are special wavefunction which are *Gaussian* and with a *fixed* spread in position and momentum, given by Eqs. (43) and (46).

models but it can be (partly) avoided with a suitable choice of the localizations operators. Within our model the mean energy reaches an asymptotic finite value, to which a temperature T can be associated; we have speculated that this temperature may be considered as the characteristic temperature of the noise, which acts on physical systems —besides localizing them in space— by thermalizing them to its temperature. Such an asymptotic value for T is quite small ($\simeq 10^{-1}$K) which implies that for typical physical systems the mean energy does not increase in time but indeed it decreases even though with a very small rate.

The price to pay in order to temper the energy non conservation is that also the momentum is not conserved, not only for single realizations of the stochastic process (as it happens for all other dynamical reduction models) but also in the average: the average momentum of any physical system slowly decays in time and asymptotically goes to zero. By the analogy with the quantum Brownian motion, the reason for this behaviour is quite simple: in order to thermalize a system to the temperature of the bath, momentum is exchanged between the system and the bath, which implies that the momentum of the quantum system alone is not conserved.

One possibility to restore momentum conservation, as well as energy conservation, would be to promote the stochastic field to a real physical field: by considering the energy and momentum of both systems, it could be possible that the two principles of energy and momentum conservation can be restored.

We conclude by noting that, whatever its nature can be, the stochastic medium cannot be quantum in the usual sense since its coupling to the particle is not a standard coupling between two quantum systems: Eq. (1), in fact, is *not* the standard Schrödinger equation with a stochastic potential.

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