Divisibility of Dynamical Maps: Schrödinger Versus Heisenberg Picture

Federico Settimo,[1,*] Andrea Smirne,[2,3] Kimmo Luoma,[1] Bassano Vacchini,[2,3] Jyrki Piilo,[1] and Dariusz Chruściński[4]

[1] *Department of Physics and Astronomy, University of Turku, FI-20014 Turun yliopisto, Finland*
[2] *Dipartimento di Fisica "Aldo Pontremoli", Università degli Studi di Milano, Via Celoria 16, I-20133 Milan, Italy*
[3] *Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, I-20133 Milan, Italy*
[4] *Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5/7, 87-100 Toruń, Poland*

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Divisibility of dynamical maps is a central notion in the study of quantum non-Markovianity, providing a natural framework to characterize memory effects via time-local master equations. In this work, we generalize the notion of divisibility of quantum dynamical maps from the Schrödinger to the Heisenberg picture. While the two pictures are equivalent at the level of physical predictions, we show that the divisibility properties of the corresponding dual maps are, in general, not equivalent. This inequivalence originates from the distinction between left and right generators of time-local master equations, which interchange roles under duality. We demonstrate that Schrödinger and Heisenberg divisibility are distinct concepts by constructing explicit dynamics divisible only in one picture. Furthermore, we introduce a quantifier for the violation of Heisenberg P-divisibility, analogous to the trace-distance-based measure of non-Markovianity, and provide it with an operational interpretation in terms of the guessing probability between effects. Our results show that Heisenberg divisibility is an independent witness of memory effects and highlight the need to consider both pictures when characterizing non-Markovian quantum dynamics.

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I. INTRODUCTION

The concept of divisibility for open quantum system dynamics has been widely investigated, in particular in connection with non-Markovianity. A quantum evolution, represented by a dynamical map $\{\Phi_t\}_{t\geq 0}$, is said to be P- or CP-divisible if the evolution between any two times s and $t > s$ can be described by positive (PTP) or completely positive trace preserving (CPTP) maps $\Phi_{t,s}$. In other words, the dynamical map at time t can be written as a composition of (C)PTP maps $\Phi_{t_i,t_{i-1}}$,

$$\Phi_t = \Phi_{t,t_n} \circ \Phi_{t_{n-1},t_{n-2}} \circ \ldots \circ \Phi_{t_2,t_1} \circ \Phi_{t_1}, \qquad (1)$$

for an arbitrary sequence $t \geq t_n \geq t_{n-1} \geq \ldots \geq t_1$.

Violations of divisibility have been connected to non-Markovianity and information backflow [1–6]. If divisibility holds, then information can only flow from the system into the environment, while its violation can be interpreted as memory effects. Other definitions of non-Markovianity based on the nonmonotonicity of distinguishability quantifiers between states [3,4,7–11] or of entanglement [12] have been proposed and some of them have been proven to be equivalent to violations of P- [3,4] or CP-divisibility [12].

Traditionally, the concept of divisibility has been only investigated in the Schrödinger picture. When the dynamics is governed by a time-local master equation (ME), divisibility can be linked to the properties of the time-dependent generator of the ME [13], and its analysis provides insights into phenomena such as information backflow and memory effects. Due to the fundamental duality between the Schrödinger and Heisenberg pictures, it is natural to ask how these concepts translate when moving from the evolution of states to the evolution of observables.

When considering semigroup dynamics [14], the corresponding generator $\mathcal{L}$ is time-independent and hence $\Phi_t = e^{t\mathcal{L}}$. Note that $\dot{\Phi}_t = \mathcal{L} \circ \Phi_t = \Phi_t \circ \mathcal{L}$ due to the fact that $\mathcal{L}$ and Φ_t commute. Hence, it is not essential whether in the ME the generator is on the left $\mathcal{L} \circ \Phi_t$ or on the right $\Phi_t \circ \mathcal{L}$. Equivalently in the Heisenberg picture $\mathcal{L}^*$ is simply the Hilbert-Schmidt adjoint of the generator $\mathcal{L}$ in

*Contact author: fesett@utu.fi

the Schrödinger picture. Again, one has $\dot{\Phi}_t^* = \mathcal{L}^* \circ \Phi_t^* = \Phi_t^* \circ \mathcal{L}^*$. However, this is no longer true beyond a semigroup scenario when the generator is time-dependent and commutativity is no longer guaranteed. Traditionally, one considers MEs $\dot{\Phi}_t = \mathcal{L}_t \circ \Phi_t$, but the map Φ_t satisfies also $\dot{\Phi}_t = \Phi_t \circ \mathcal{R}_t$ with an appropriate generator $\mathcal{R}_t$ that is generally different from $\mathcal{L}_t$, $\mathcal{R}_t \neq \mathcal{L}_t$. In this paper we propose to call a *traditional* $\mathcal{L}_t$ a *left generator* and $\mathcal{R}_t$ a *right generator*. Of course, in the commutative case, i.e., when $\mathcal{L}_t$ commutes with Φ_t, left and right generators coincide, $\mathcal{L}_t = \mathcal{R}_t$.

The meaning of the left generator $\mathcal{L}_t$ is clear. In particular it does control the divisibility property of the dynamical map Φ_t in the Schrödinger picture. What is then the meaning of the right generator $\mathcal{R}_t$? In this work, we generalize the concept of divisibility to the Heisenberg picture and show that $\mathcal{R}_t^*$ controls the divisibility property of the dynamical map Φ_t^* in the Heisenberg picture. Our key result is that traditional Schrödinger divisibility and Heisenberg divisibility are not equivalent, i.e., a dynamical map Φ_t may be Schrödinger divisible whereas Φ_t^* may be Heisenberg indivisible and vice versa. Furthermore, we provide a quantifier of the violation of P-divisibility in the Heisenberg picture analogous to the measure of non-Markovianity proposed in [7,8], connected to nonmonotonicity of distinguishability quantifiers between effects.

The rest of the paper is organized as follows. In Sec. II, we characterize the left and the right generators, both in the Schrödinger and in the Heisenberg picture, showing that a left generator becomes a right one when going from one picture to the other. In Sec. III, we explore the concept of divisibility in both pictures, showing that in the Heisenberg picture it is generated by the *right* generator of the Schrödinger master equation. We therefore conclude that divisibility in the two pictures are not equivalent and propose a measure of violations of divisibility in the Heisenberg picture. In Sec. IV, we study Heisenberg divisibility in connection with dynamical evolution of positive operator-valued measures (POVMs), showing that a nonmonotonic behavior of either compatibility or sharpness can be used to witness violations of divisibility. In Sec. V, we propose an explicit example of qubit dynamics divisible only in one picture but not in the other. Finally, in Sec. VI, we provide an overview of our work and point to future developments.

II. LEFT AND RIGHT GENERATORS OF THE MASTER EQUATION

We now introduce the left and right generators for open quantum system dynamics, in both the Schrödinger and the Heisenberg pictures.

A. Schrödinger picture

In open quantum systems, under the assumption that the open system is initially uncorrelated with its environment, the dynamics of the reduced density matrix is described by a CPTP dynamical map $\Phi_t : \rho \mapsto \Phi_t[\rho] = \rho(t)$ [15–18]. Assuming that the dynamics is invertible, i.e., that Φ_t^{-1} exists at all times, then Φ_t is guaranteed to be the solution of the time-local ME

$$\dot{\Phi}_t = \mathcal{L}_t \circ \Phi_t, \tag{2}$$

which, in turn, implies the following ME for the density operator:

$$\frac{d}{dt}\rho(t) = \mathcal{L}_t[\rho(t)]. \tag{3}$$

The time-dependent generator $\mathcal{L}_t = \dot{\Phi}_t \circ \Phi_t^{-1}$ has the following universal structure [19,20]:

$$\mathcal{L}_t[\rho] = -i[H(t),\rho] + \sum_\alpha \gamma_\alpha(t)\left[L_\alpha(t)\rho L_\alpha^\dagger(t) - \frac{1}{2}\{L_\alpha^\dagger(t)L_\alpha(t),\rho\}\right], \tag{4}$$

where $H(t) = H^\dagger(t)$, which follows from the requirements that the dynamics is trace- and Hermiticity-preserving, in turn implying that the generator is trace-annihilating and Hermiticity-preserving, i.e., $\mathrm{tr}\mathcal{L}_t[\rho] = 0$ and $\mathcal{L}_t[\rho]^\dagger = \mathcal{L}_t[\rho]$. The solution of the ME (2) can be written as

$$\Phi_t = T_\leftarrow \exp\left(\int_0^t d\tau \mathcal{L}_\tau\right), \tag{5}$$

where $T_\leftarrow$ is the chronological time ordering.

Equivalently, the ME can also be written as

$$\dot{\Phi}_t = \Phi_t \circ \mathcal{R}_t, \tag{6}$$

with an appropriate generator $\mathcal{R}_t$. Indeed, if one knows Φ_t, then one can simply define $\mathcal{R}_t := \Phi_t^{-1} \circ \dot{\Phi}_t$. Note that $\mathcal{R}_t$ has the same form as $\mathcal{L}_t$ since it is trace-annihilating and Hermiticity-preserving as well, and therefore it can be represented as

$$\mathcal{R}_t[\rho] = -i[K(t),\rho] + \sum_\alpha \xi_\alpha(t)\left[R_\alpha(t)\rho R_\alpha^\dagger(t) - \frac{1}{2}\left\{R_\alpha^\dagger(t)R_\alpha(t),\rho\right\}\right], \tag{7}$$

with $K^\dagger(t) = K(t)$. Therefore, the dynamical map Φ_t can alternatively be written as

$$\Phi_t = T_\rightarrow \exp\left(\int_0^t d\tau \mathcal{R}_\tau\right), \tag{8}$$

where $T_{\rightarrow}$ is the antichronological time ordering. On the other hand, (6) does not imply a ME for $\rho(t)$, but one has

$$\frac{d}{dt}\rho(t) = \Phi_t\left[\mathcal{R}_t[\rho(0)]\right], \tag{9}$$

which requires not only the knowledge of $\mathcal{R}_t$ but also of the map Φ_t. However, knowing the map, one can immediately generate the corresponding state $\rho(t) = \Phi_t[\rho(0)]$ starting from the arbitrary initial state $\rho(0)$. This is the main reason why we traditionally use $\mathcal{L}_t$ but not $\mathcal{R}_t$: Knowing $\mathcal{L}_t$ and an initial state $\rho(0)$, we can find $\rho(t)$ by directly solving Eq. (4).

In the following, we refer to MEs of the form (2) as *left ME* and to its generator $\mathcal{L}_t$ as *left generator*. Similarly, we refer to MEs of the form (6) as *right ME* and to generators $\mathcal{R}_t$ as *right generator*. Naturally, it is possible to derive the left generator from the right one and vice versa (once the map Φ_t is known) as

$$\mathcal{R}_t = \Phi_t^{-1}\circ\mathcal{L}_t\circ\Phi_t, \quad \mathcal{L}_t = \Phi_t\circ\mathcal{R}_t\circ\Phi_t^{-1}. \tag{10}$$

By comparing Eqs. (5) and (8), it is easy to see that in general $\mathcal{R}_t \neq \mathcal{L}_t$, unless

$$[\mathcal{L}_t, \mathcal{L}_{t'}] = 0 \tag{11}$$

for all times t, t'. This, in turns, implies $[\mathcal{L}_t, \Phi_t] = 0$ and using Eq. (10) it implies $\mathcal{L}_t = \mathcal{R}_t$. The commutativity of $\mathcal{L}_t$ occurs, for example, if the dynamics is a semigroup, for which $\mathcal{L}_t$ does not depend on time, or if only one jump operator is present with $[L(t), L(t')] = 0$.

If, as a special case, one considers a pure unitary (i.e., closed system) dynamics with the left generator $\mathcal{L}_t = -i[H(t), \cdot]$, then the right generator is the commutator with the Hamiltonian time-evolved in the Heisenberg picture, i.e.,

$$\mathcal{R}_t = -i\left[U^\dagger(t)H(t)U(t), \cdot\right], \tag{12}$$

where $U(t)$ is the unitary evolution generated by the Hamiltonian $H(t)$.

To further illustrate the intricate relation between $\mathcal{L}_t$ and $\mathcal{R}_t$, it is convenient to consider the exponential representation of the map Φ_t,

$$\Phi_t = e^{G_t}, \tag{13}$$

with some map G_t and the exponential is taken without time-ordering. Clearly $G_{t=0} = 0$, and let $\mathcal{M}_t = \dot{G}_t$, i.e., $G_t = \int_0^t \mathcal{M}_\tau d\tau$; using the well-known Wilcox formula [21], one has

$$\dot{\Phi}_t = \frac{d}{dt}e^{G_t} = \int_0^1 e^{sG_t}\mathcal{M}_t e^{(1-s)G_t}ds = \mathcal{L}_t\circ\Phi_t, \tag{14}$$

where

$$\mathcal{L}_t := \int_0^1 e^{sG_t}\mathcal{M}_t e^{-sG_t}ds. \tag{15}$$

Similarly, by changing $s \to 1-s$, one gets from (14)

$$\dot{\Phi}_t = \int_0^1 e^{(1-s)G_t}\mathcal{M}_t e^{sG_t}\,ds = \Phi_t\circ\mathcal{R}_t, \tag{16}$$

where

$$\mathcal{R}_t := \int_0^1 e^{-sG_t}\mathcal{M}_t e^{sG_t}ds. \tag{17}$$

It is therefore clear that if $[G_t, \dot{G}_{t'}] = 0$ or, equivalently,

$$\left[\int_0^t \mathcal{M}_\tau d\tau, \mathcal{M}_{t'}\right] = 0, \tag{18}$$

then

$$\mathcal{L}_t = \mathcal{M}_t = \mathcal{R}_t. \tag{19}$$

Otherwise, $\mathcal{L}_t$ and $\mathcal{R}_t$ are different. We stress that in general $\mathcal{M}_t$ is not a generator for the dynamical map Φ_t in the sense that Φ_t is not a solution of the ME generated by $\mathcal{M}_t$. Only if commutativity (18) holds it becomes a legitimate time-local generator for Φ_t and it coincides with the left and right generators.

B. Heisenberg picture

By duality, one can define the dynamical map in the Heisenberg picture Φ_t^* via

$$\mathrm{tr}\left[X\,\Phi_t[\rho]\right] = \mathrm{tr}\left[\Phi_t^*[X]\,\rho\right] \quad \forall\rho, X, \tag{20}$$

where Φ_t^* is the Hilbert-Schmidt adjoint of Φ_t and it is a completely positive unital (CPU) map $\Phi_t^*[\mathbb{1}] = \mathbb{1}$. If Φ_t is unital, then Φ_t^* is also trace preserving and vice versa.

The dual of the ME (2) and (6) reads

$$\dot{\Phi}_t^* = \Phi_t^*\circ\mathcal{L}_t^* = \mathcal{R}_t^*\circ\Phi_t^*, \tag{21}$$

where $\mathcal{L}_t^*$ and $\mathcal{R}_t^*$ are the Hilbert-Schmidt adjoints of $\mathcal{L}_t$ and $\mathcal{R}_t$, respectively,

$$\mathcal{R}_t^* = \dot{\Phi}_t^*\circ\Phi_t^{*-1}, \quad \mathcal{L}_t^* = \Phi_t^{*-1}\circ\dot{\Phi}_t^*. \tag{22}$$

Notice that $\mathcal{L}_t$ generates the left ME (2), while its dual $\mathcal{L}_t^*$ generates the right ME in the Heisenberg picture. The

Heisenberg left ME is instead generated by $\mathcal{R}_t^*$, i.e., the dual of the right generator in the Schrödinger picture. Equivalently, the ME (21) can be written as

$$\frac{d}{dt}X(t) = \Phi_t^*\left[\mathcal{L}_t^*[X(0)]\right] = \mathcal{R}_t^*\left[X(t)\right]. \tag{23}$$

Notice that in the Heisenberg picture $\mathcal{R}_t^*$ generates the ME for $X(t)$, while in the Schrödinger picture the ME for the density matrix $\rho(t)$ is generated by $\mathcal{L}_t$.

The generators can be written explicitly as

$$\mathcal{R}_t^*[X] = i[K(t), X] + \sum_\alpha \xi_\alpha(t)\left[R_\alpha^\dagger(t) X R_\alpha(t) - \frac{1}{2}\left\{R_\alpha^\dagger(t)R_\alpha(t), X\right\}\right], \tag{24}$$

and similarly for $\mathcal{L}_t^*$. It is possible to obtain $\mathcal{R}_t^*$ from $\mathcal{L}_t^*$ via the dual of Eq. (10)

$$\mathcal{R}_t^* = \Phi_t^* \circ \mathcal{L}_t^* \circ \Phi_t^{*-1}, \quad \mathcal{L}_t^* = \Phi_t^{*-1} \circ \mathcal{R}_t^* \circ \Phi_t^*. \tag{25}$$

Similarly to Eqs. (5) and (8), the solution of the dual ME (21) can be written as either a chronological or an antichronological time-ordered exponential

$$\Phi_t^* = T_\rightarrow \exp\left(\int_0^t d\tau\, \mathcal{L}_\tau^*\right) = T_\leftarrow \exp\left(\int_0^t d\tau\, \mathcal{R}_\tau^*\right). \tag{26}$$

Notice that when considering the dual, chronological, and antichronological time ordering is exchanged: in the Schrödinger picture $\mathcal{L}_t$ appears in a time-ordered exponential, while its dual $\mathcal{L}_t^*$ in an antitime-ordered exponential and vice versa. This follows from the fact that $(AB)^* = B^*A^*$ for arbitrary operators A and B.

III. DIVISIBILITY CONDITIONS

In the Schrödinger picture, the concept of divisibility has been widely studied and is connected to the properties of the left generator $\mathcal{L}_t$. In this section, we first briefly recall the definition of divisibility and its connection with non-Markovianity. We then generalize this concept to the Heisenberg picture and assign an operational interpretation to violations of divisibility also in this picture.

A. Divisibility in the Schrödinger picture

If the dynamical map Φ_t is invertible, it is possible to define a two-parameter propagator

$$\Phi_{t,s}^L = \Phi_t \circ \Phi_s^{-1} = T_\leftarrow \exp\left(\int_s^t d\tau \mathcal{L}_\tau\right), \tag{27}$$

describing the evolution from a time s to a later time $t > s$.

In a similar manner, it is possible to define a different two-parameter propagator starting from the right generator

$$\Phi_{t,s}^R = \Phi_s^{-1} \circ \Phi_t = T_\rightarrow \exp\left(\int_s^t d\tau \mathcal{R}_\tau\right), \tag{28}$$

such that the dynamical map at time t can be obtained in terms of Φ_s in two ways,

$$\Phi_t = \Phi_{t,s}^L \circ \Phi_s = \Phi_s \circ \Phi_{t,s}^R. \tag{29}$$

We call $\Phi_{t,s}^L$ the *left propagator* and $\Phi_{t,s}^R$ the *right propagator*. Of the two propagators, only $\Phi_{t,s}^L$ describes the time evolution of the state for intermediate times

$$\rho(t) = \Phi_{t,s}^L[\rho(s)], \tag{30}$$

while for $\Phi_{t,s}^R$ one has

$$\rho(t) = \Phi_s\left[\Phi_{t,s}^R\left[\rho(0)\right]\right]. \tag{31}$$

This is particularly useful if one wants to computationally simulate the dynamical map: given the state $\rho(t)$ at some time t, the state after an infinitesimal time increment dt is

$$\rho(t+dt) = \Phi_{t+dt,t}^L[\rho(t)] \approx \rho(t) + dt\,\mathcal{L}_t[\rho(t)]. \tag{32}$$

Therefore, the infinitesimal time evolution is obtained via the left generator $\mathcal{L}_t$.

The dynamics is said to be (C)P-divisible if the left propagator $\Phi_{t,s}^L$ is (C)PTP for all times $t \geq s$. CP-divisibility corresponds to non-negativity of all of the rates $\gamma_\alpha(t) \geq 0$ of the left generator $\mathcal{L}_t$, while P-divisibility is equivalent to [13]

$$\sum_\alpha \gamma_\alpha(t)\left|\langle\varphi_\mu|L_\alpha(t)|\varphi_{\mu'}\rangle\right|^2 \geq 0, \tag{33}$$

for all orthonormal bases $\{\varphi_\mu\}_\mu$ and for all $\mu \neq \mu'$.

Divisibility has been widely studied in the literature, and its violations have been connected to non-Markovianity of the dynamics [1,2,7,8,12]. In particular, it is possible to give an operational interpretation to violations of P-divisibility as memory effects in the following way. Suppose that Alice prepares one of two states ρ or σ randomly, each with probability 1/2, and sends it to Bob whose task is to guess which state was prepared. Bob's probability of correctly guessing by making a single measurement is given by [22]

$$P_{\text{guess}}^{\text{s}}(\rho,\sigma) = \frac{1}{2}(1 + D_1(\rho,\sigma)), \tag{34}$$

where D_1 is the trace distance [23]

$$D_1(\rho,\sigma) := \frac{1}{2}\left\|\rho - \sigma\right\|_1, \quad \|X\|_1 := \text{tr}\,|X|. \tag{35}$$

Because of the contractivity of the trace distance under CPTP maps, then one has that the dynamics cannot increase P^{s}_{guess} above its initial value

$$P^{s}_{guess}\left(\Phi_t[\rho],\Phi_t[\sigma]\right)\le P^{s}_{guess}(\rho,\sigma). \quad (36)$$

However, revivals in time of P^{s}_{guess} can be present. Notice that since the dynamical map Φ_t is P-divisible if and only if [3,4,13]

$$\frac{d}{dt}\|\Phi_t[X]\|_1\le 0, \quad (37)$$

for all self-adjoint operators X, then revivals can be present only if P-divisibility is violated. Such revivals can be interpreted as memory effects due to the presence of the environment. It is possible to bound the revival from a time s to a later time $t\ge s$ as [9,10,24]

$$\begin{aligned}D_1\left(\rho_S^1(t),\rho_S^2(t)\right)-D_1\left(\rho_S^1(s),\rho_S^2(s)\right)&\le D_1\left(\rho_E^1(s),\rho_E^2(s)\right)\\&+D_1\left(\rho_{SE}^1(s),\rho_S^1(s)\otimes\rho_E^1(s)\right)\\&+D_1\left(\rho_{SE}^2(s),\rho_S^2(s)\otimes\rho_E^2(s)\right). \quad (38)\end{aligned}$$

In order for a revival to be present, either correlations between the system and the environment are present at time s or the environmental states have become different. It is possible to quantify the degree of non-Markovianity as [7,8]

$$\mathcal{N}_S(\Phi):=\sup_{\rho,\sigma}\int_{\dot\eta_S(t)>0}dt\,\dot\eta_S(t), \quad (39)$$

where $\eta_S(t):=D_1(\Phi_t[\rho],\Phi_t[\sigma])$ and the supremum is taken over all pairs of states. If $\mathcal{N}_S(\Phi)>0$ then the dynamics must necessarily be non P-divisible.

Notice that the concept of divisibility can be extended also to dynamics which are not invertible [25,26] by using the generalized inverse [27] which is always guaranteed to exist even when Φ_t^{-1} does not exist. In this work, however, we will limit ourselves to the invertible case, as is typically done with divisibility and in the framework of operational interpretation.

B. Divisibility in the Heisenberg picture

Similar to Φ_t, also the dynamical map in the Heisenberg picture Φ_t^* can be written in terms of the map at the intermediate time s in two ways,

$$\Phi_t^*=\Phi_s^*\circ\Phi_{t,s}^{L*}=\Phi_{t,s}^{R*}\circ\Phi_s^*, \quad (40)$$

where $\Phi_{t,s}^{R*}$ and $\Phi_{t,s}^{L*}$ are both unital maps, reading

$$\Phi_{t,s}^{R*}=\Phi_t^*\circ\Phi_s^{*-1}=T_{\leftarrow}\exp\left(\int_s^t d\tau\,\mathcal{R}_\tau^*\right) \quad (41)$$

and

$$\Phi_{t,s}^{L*}=\Phi_s^{*-1}\circ\Phi_t^*=T_{\rightarrow}\exp\left(\int_s^t d\tau\,\mathcal{L}_\tau^*\right). \quad (42)$$

Once again chronological and antichronological time orderings are swapped with respect to the Schrödinger picture. Accordingly, the dual of the left propagator $\Phi_{t,s}^{L}$ is now a right propagator $\Phi_{t,s}^{L*}$ and vice versa. This means that the evolution from the intermediate time s to the later time t is now described by

$$X(t)=\Phi_{t,s}^{R*}[X(s)], \quad (43)$$

and, analogously to Eq. (32), the infinitesimal evolution in the Heisenberg picture reads

$$X(t+dt)=\Phi_{t+dt,t}^{R*}[X(t)]\approx X(t)+dt\,\mathcal{R}_t^*[X(t)]. \quad (44)$$

It is worth stressing that, although in the Schrödinger picture the intermediate evolution is generated by the left generator $\mathcal{L}_t$, in the Heisenberg picture it is *not* obtained from its dual, but from the dual of the right generator $\mathcal{R}_t^*$, which becomes a left generator in the Heisenberg picture.

From Eq. (43), it is natural to define the concept of divisibility also in the Heisenberg picture: The dynamics is said to be *Heisenberg (C)P-divisible* if $\Phi_{t,s}^{R*}$ is (C)PU for all times $t\ge s$. Because of duality, Heisenberg divisibility is equivalent to (complete) positivity of the right propagator $\Phi_{t,s}^{R}$ of Eq. (28) and not of the left propagator $\Phi_{t,s}^{L}$ which determines Schrödinger divisibility. Therefore, divisibility in the Schrödinger and in the Heisenberg pictures are two nonequivalent concepts. Heisenberg CP-divisibility is equivalent to the positivity of the rates $\xi_\alpha(t)$ of the right generator $\mathcal{R}_t$ of Eq. (6), while for P-divisibility an analogous of Eq. (33) holds, but using the rates $\xi_\alpha(t)$ and operators $R_\alpha(t)$ of the right generator.

Notice that, in the special case of unitary evolution in one picture, the dynamics is unitary also in the other picture, as shown in Eq. (12). Therefore, for unitaries, divisibility is preserved when going from one picture to the other.

1. Operational interpretation

We now proceed to show that it is possible to give an interpretation to violations of Heisenberg P-divisibility analogous to the revivals of the guessing probability of Eq. (34) that holds in the Schrödinger picture. Consider a scenario dual to the one considered for Eq. (34): Alice prepares a black box performing one of two possible measurements, each with the same probability and described by effects E and F. Bob's task is to guess what measurement is being performed. To do so, he can prepare a single

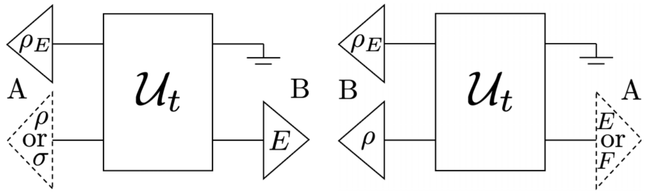

FIG. 1. Left: guessing between states (Schrödinger divisibility): Alice prepares either ρ or σ and Bob has to guess which state was prepared. Right: guessing between effects (Heisenberg divisibility): Alice can measure either E or F and Bob has to guess which effect was measured.

state ρ and use the black box only once. In Appendix A we show that Bob's probability of correctly guessing is

$$P^{\mathrm{e}}_{\mathrm{guess}}(E,F) = \frac{1}{2}\left(1 + D_\infty(E,F)\right), \tag{45}$$

where D_∞ is the operator distance [28]

$$D_\infty(E,F) := \|E - F\|_\infty, \tag{46}$$

defined from the operator norm

$$\begin{aligned}\|X\|_\infty &:= \max\{|\lambda_\alpha|, \lambda_\alpha \text{ eigenvalue of } X\} \\ &= \max_\psi |\langle\psi|X|\psi\rangle|.\end{aligned} \tag{47}$$

The operator norm D_∞ therefore represents the bias in favor of correctly guessing which effect was measured by Alice, just like D_1 represents the bias in favor of guessing which state was prepared. Since $-\mathbb{1} \le E - F \le \mathbb{1}$, the operator distance is bounded $0 \le D_\infty(E,F) \le 1$, and then $P^{\mathrm{e}}_{\mathrm{guess}}(E,F)$ is indeed a probability. See Fig. 1 for a representation of the guessing probabilities in the Schrödinger and Heisenberg pictures.

Revivals of such guessing probability can be linked to violations of divisibility. In fact, a unital map Λ is positive if and only if [29]

$$\|\Lambda(X)\|_\infty \le \|X\|_\infty, \tag{48}$$

for all operators X. In analogy with Eq. (37), it is possible to derive a condition for divisibility also in the Heisenberg picture: a CPU dynamical map Φ^*_t is Heisenberg P-divisible if and only if

$$\frac{d}{dt}\left\|\Phi^*_t[X]\right\|_\infty \le 0. \tag{49}$$

Therefore, the operator distance D_∞ presents revivals if and only if Heisenberg P-divisibility is violated. It is possible to bound the revival of the distance between two effects from a time s to a later time $t \ge s$ as

$$\begin{aligned}D_\infty\left(X^1_S(t), X^2_S(t)\right) - D_\infty\left(X^1_S(s), X^2_S(s)\right) &\le D_\infty\left(X^1_E(s), X^2_E(s)\right) \\ &+ D_\infty\left(X^1_{SE}(s), X^1_S(s)\otimes X^1_E(s)\right) \\ &+ D_\infty\left(X^2_{SE}(s), X^2_S(s)\otimes X^2_E(s)\right),\end{aligned} \tag{50}$$

where $X^i_{SE}(0) = X^i_S(0)\otimes\mathbb{1}_E$ with $\{X^i_S(0)\}_{i=1,2}$ two distinct effects, $X^i_{SE}(t)$ are the unitarily evolved Heisenberg picture operators, while $X^i_S(t)$ and $X^i_E(t)$ are the effects evolved according to the reduced dynamics, that is $X^i_S(t) = \mathrm{tr}_E\{\rho_E X^i_{SE}(t)\}$ as well as $X^i_E(t) = \mathrm{tr}_S\{\rho_S X^i_{SE}(t)\}$. As discussed in more detail in Appendix B where the result is proven, this bound is dual to the bound of revivals of D_1 of Eq. (38).

Using Eqs. (45) and (49) it is thus possible to give an operational interpretation of Heisenberg P-divisibility: if Φ^*_t is P-divisible, then the probability of guessing which effect was chosen by Alice will decrease monotonically in time—information is lost to the environment. If, instead, P-divisibility is violated, then there exist two effects E and F whose guessing probability is not monotonic. Equation (50) allows us to interpret this revival as a backflow of information: the information initially lost in the environment can be stored as correlations (via $D_\infty(X^i_{SE}(s), X^i_S(s)\otimes X^i_E(s)) \ne 0$) or differences in the environments (via $D_\infty(X^1_E(s), X^2_E(s)) \ne 0$) and can later flow back into the open system at later times $t \ge s$.

In a similar manner to what was done in the Schrödinger picture via Eq. (39), it is possible to quantify the violation of Heisenberg P-divisibility as

$$\mathcal{N}_H(\Phi^*) := \sup_{E,F}\int_{\dot\eta_H(t)>0} dt\,\dot\eta_H(t), \tag{51}$$

where $\eta_H(t) := D_\infty(\Phi^*_t[E], \Phi^*_t[F])$ and the sup is taken over all pairs of effects $0 \le E, F \le \mathbb{1}$. From Eq. (49), it holds that $\mathcal{N}_H(\Phi^*) > 0$ if and only if the dynamics is Heisenberg P-divisible.

Notice that revivals of the operator distance D_∞ were considered in the Schrödinger picture [28]. However, since contractivity is guaranteed only for unital maps, revivals under completely positive but nonunital maps were interpreted as a witness of nonclassicality of the dynamics [30]. Here, instead, since we are considering dynamics in the Heisenberg picture, unitality, and thus contractivity under completely positive maps, is always guaranteed to hold. Notice also that violations of unitality in the Schrödinger picture are equivalent to violations of trace preservation in the Heisenberg picture and can be interpreted as adding biases to the measurement apparatus.

C. Example: Phase covariant dynamics

As an example showing the different forms for the left and the right generators, as well as the different conditions for divisibility, let us consider the phase covariant dynamics defined by [31–33]

$$\Phi_t[\rho] = \frac{1}{2}\left[\mathrm{tr}[\rho](\mathbb{1} + \lambda_T\sigma_z) + \lambda_z \mathrm{tr}[\rho\sigma_z]\sigma_z \right. \\ \left. +\lambda\left(\mathrm{tr}[\rho\sigma_x]\sigma_x + \mathrm{tr}[\rho\sigma_y]\sigma_y\right)\right], \tag{52}$$

where λ_T, λ, and λ_z are time-dependent real functions such that

$$|\lambda_T| + |\lambda_z| \le 1, \quad 4\lambda^2 + \lambda_T^2 \le (1+\lambda_z)^2 \tag{53}$$

and σ_i are the Pauli matrices. The left generator reads

$$\mathcal{L}_t[\rho] = \sum_{\alpha=z,\pm} \gamma_\alpha \left[\sigma_\alpha\rho\sigma_\alpha^\dagger - \frac{1}{2}\left\{\sigma_\alpha^\dagger\sigma_\alpha, \rho\right\}\right], \tag{54}$$

where $\sigma_+ = |1\rangle\langle 0| = \sigma_-^\dagger$ are the raising and lowering operators, with rates

$$\gamma_\pm = \frac{\pm\dot{\lambda}_T\lambda_z - (1+\lambda_T)\dot{\lambda}_z}{2\lambda_z}, \\ \gamma_z = \frac{1}{4}\left(\frac{\dot{\lambda}_z}{\lambda_z} - 2\frac{\dot{\lambda}}{\lambda}\right). \tag{55}$$

In the Schrödinger picture, the dynamics is CP-divisible if and only if all the rates are positive $\gamma_\alpha \ge 0$, while it is P-divisible if and only if [34,35]

$$\gamma_\pm \ge 0 \quad \text{and} \quad \gamma_z \ge -\frac{1}{2}\sqrt{\gamma_+\gamma_-}. \tag{56}$$

It is easy to show that the right generator $\mathcal{R}_t = \Phi_t^{-1} \circ \dot{\Phi}_t$ has the same functional form of the left generator (54) with rates

$$\xi_\pm = \frac{\pm\dot{\lambda}_T - \dot{\lambda}_z}{2\lambda_z}, \quad \xi_z = \gamma_z. \tag{57}$$

Interestingly, the rates corresponding to σ_z are equal ($\gamma_z = \xi_z$), while the other two rates are in general different, and therefore the left and the right generators do not coincide. It is possible to write the right rates in terms of the left ones as

$$\xi_\pm = \frac{\gamma_+(\lambda_z \mp \lambda_T \pm 1) + \gamma_-(\lambda_z \mp \lambda_T \mp 1)}{2\lambda_z}. \tag{58}$$

Notice that when the dynamics is unital ($\lambda_T = 0$ or, equivalently, $\gamma_+ = \gamma_-$), then the dynamics is commutative and therefore $\mathcal{L}_t = \mathcal{R}_t$.

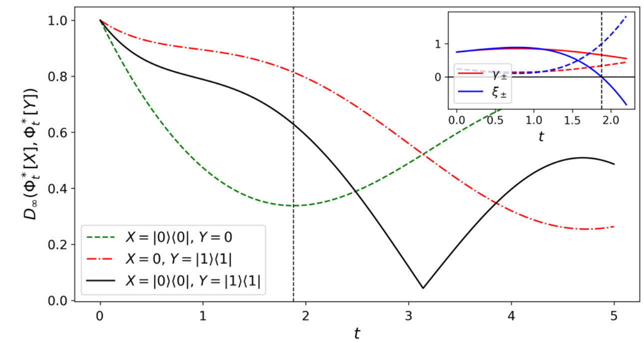

FIG. 2. Phase covariant dynamics. Dynamics of $D_\infty(\Phi_t^*[X], \Phi_t^*[Y])$ for different initial effects X, Y. Inset: rates γ_+ (red solid), γ_- (red dashed), ξ_+ (blue solid), and ξ_- (blue dashed) corresponding to Eq. (59). At $t \approx 1.9$ (vertical dashed line), ξ_+ becomes negative and, accordingly, D_∞ behaves nonmonotonically, thus witnessing violations of Heisenberg P-divisibility.

The conditions for Heisenberg P- and CP-divisibility are the same as the Schrödinger ones but with ξ_α instead of γ_α. However, due to the fact that $\gamma_\pm \ne \xi_\pm$, the two divisibilities are not equivalent. In particular, it is possible to have Schrödinger CP-divisible but Heisenberg P-indivisible dynamics. An explicit example is given by the choice

$$\lambda_T(t) = \frac{1}{2}\sin t, \quad \lambda_z(t) = e^{-t}, \quad \lambda(t) = e^{-t/2}, \tag{59}$$

for which $\gamma_z = \xi_z = 0$, $\gamma_\pm \ge 0$ but $\xi_\pm$ are not positive. Such rates are shown in the inset of Fig. 2. The nonpositivity of the right rates implies that the dynamics in the Heisenberg picture is non-P-divisible. This fact is reflected by a nonmonotonicity of $D_\infty(\Phi_t^*[E], \Phi_t^*[F])$ for some effects E, F as a consequence of Eq. (49) and is shown in Fig. 2. The code used for obtaining the plots is available at [36].

The other way around, instead, is not possible: if the dynamics is Heisenberg CP-divisible (and invertible) then it is also Schrödinger CP-divisible. In fact, the condition $\xi_\pm \ge 0$ implies

$$\dot{\lambda}_z \le \min\{\dot{\lambda}_T, -\dot{\lambda}_T\} = -\left|\dot{\lambda}_T\right|. \tag{60}$$

Substituting it in Eq. (55) and using $\lambda_z > 0$ (because of invertibility), then

$$2\lambda_z\gamma_\pm \ge \left|\dot{\lambda}_T\right|\left(1 \pm \lambda_T + \lambda_z\,\mathrm{sgn}(\dot{\lambda}_T)\right), \tag{61}$$

where $\mathrm{sgn}(x) = x/|x|$ is the sign function. Due to the CP conditions of (53), the term within the parentheses is always positive and therefore $\gamma_\pm \ge 0$.

The fact that right divisibility implies left divisibility is specific to the particular model chosen and does not hold in general. In Sec. V B we provide explicit examples of dynamics either Heisenberg or Schrödinger divisible.

D. Classical stochastic dynamics

Similar considerations also hold in the classical setting. The dynamics of a d-site probability distribution $\mathbf{p}(t) = S(t)\mathbf{p}(0)$, where $\{S(t)\}_{t\geq 0}$ is a classical dynamical map represented by $d \times d$ stochastic matrix, obeys the MEs

$$\dot{S}(t) = L(t)S(t), \quad \dot{S}(t) = S(t)R(t), \tag{62}$$

where $L(t)$ and $R(t)$ are, respectively, the left and right generators and are also $d \times d$ matrices. Recall that $S(t)$, being stochastic, satisfies

$$S_{ij}(t) \geq 0, \quad \sum_{i=1}^{d} S_{ij}(t) = 1, \tag{63}$$

and $S_{ij}(t=0) = \delta_{ij}$. For the dynamics to preserve the total probability, the corresponding right and left generators must satisfy

$$\sum_{i=1}^{d} L_{ij}(t) = 0, \quad \sum_{i=1}^{d} R_{ij}(t) = 0. \tag{64}$$

Similarly to the quantum case, the solution can be written as

$$S(t) = T_{\leftarrow} \exp\left(\int_0^t d\tau\, L(\tau)\right) = T_{\rightarrow} \exp\left(\int_0^t d\tau\, R(\tau)\right) \tag{65}$$

and the generators can be written as

$$L(t) = \dot{S}(t)S(t)^{-1}, \quad R(t) = S(t)^{-1}\dot{S}(t). \tag{66}$$

It is possible to derive the right generator from the left one as

$$R(t) = S(t)^{-1}L(t)S(t), \tag{67}$$

$$L(t) = S(t)R(t)S(t)^{-1}. \tag{68}$$

Also in the classical case, the two generators will coincide if and only if the dynamics is commutative, i.e., $[L(t), L(t')] = 0$.

Notice that one can reformulate (62) as a ME for $\mathbf{p}(t)$

$$\dot{\mathbf{p}}(t) = L(t)\mathbf{p}(t), \tag{69}$$

however, there is no corresponding ME for $\mathbf{p}(t)$ in terms of the right generator $R(t)$. This is why traditionally one prefers to use the left one $L(t)$. Moving to the "Heisenberg" dual picture, one arrives at the ME for a classical observable represented by a vector $\mathbf{x} \in \mathbb{R}^d$

$$\dot{\mathbf{x}}(t) = R^{\top}(t)\mathbf{x}(t), \tag{70}$$

where the $R^{\top}$ stands for the transposed matrix. Also in this case, the ME in the dual Heisenberg pictures is defined by $R^{\top}(t)$ and not by $L^{\top}(t)$. One finds

$$(\mathbf{x}(0), \mathbf{p}(t)) = (\mathbf{x}(t), \mathbf{p}(0)), \tag{71}$$

where $(\mathbf{a}, \mathbf{b}) = \sum_k a_k b_k$, and hence

$$(\mathbf{x}(0), L(t)\mathbf{p}(t)) = (R(t)\mathbf{x}(t), \mathbf{p}(0)). \tag{72}$$

The classical map $S(t)$ is Markovian [37], meaning P-divisible, if it is possible to write it as

$$S(t) = S^L(t,u)S(u), \tag{73}$$

and the $S^L(t,u)$ is stochastic for all $t \geq u$. We call $S^L(t,u)$ a left propagator. This property is fully controlled by the left generator $L(t)$.

Proposition 1. The classical dynamical map $\{S(t)\}_{t\geq 0}$ is P-divisible if $L(t)$ satisfies Kolmogorov conditions, i.e.,

$$L_{ij}(t) \geq 0, \quad (i \neq j), \tag{74}$$

for all $t \geq 0$.

It is clear that $S^L(t,u) = S(t)S^{-1}(u)$. Similarly, one may define the right propagator $S^R(t,u) = S^{-1}(u)S(t)$ such that

$$S(t) = S(u)S^R(t,u). \tag{75}$$

One has

$$S^L(t,u) = T_{\leftarrow} \exp\left(\int_u^t d\tau\, L(\tau)\right), \tag{76}$$

$$S^R(t,u) = T_{\rightarrow} \exp\left(\int_u^t d\tau\, R(\tau)\right). \tag{77}$$

Notice that the positivity of the off-diagonal elements of $L_{ij}(t)$ does not imply positivity of the off-diagonal elements of $R_{ij}(t)$.

Example 1. Let us illustrate the relation between $L(t)$ and $R(t)$ for a two-state system. A 2×2 stochastic matrix can be represented as

$$S(t) = \begin{pmatrix} a(t) & 1-b(t) \\ 1-a(t) & b(t) \end{pmatrix}, \tag{78}$$

with $0 \le a(t), b(t) \le 1$. The generators $L(t)$ and $R(t)$ are given by

$$L(t) = \dot{S}(t)S^{-1}(t) = \begin{pmatrix} -\ell_1(t) & \ell_2(t) \\ \ell_1(t) & -\ell_2(t) \end{pmatrix}, \tag{79}$$

$$R(t) = S(t)^{-1}\dot{S}(t) = \begin{pmatrix} -r_1(t) & r_2(t) \\ r_1(t) & -r_2(t) \end{pmatrix}, \tag{80}$$

where

$$\ell_1(t) = \frac{-w(t) - \dot{b}(t)}{a(t)+b(t)-1}, \tag{81}$$

$$\ell_2(t) = \frac{w(t) - \dot{a}(t)}{a(t)+b(t)-1}, \tag{82}$$

$$r_1(t) = \frac{-\dot{a}(t)}{a(t)+b(t)-1}, \tag{83}$$

$$r_2(t) = \frac{-\dot{b}(t)}{a(t)+b(t)-1}, \tag{84}$$

together with $w(t) = \dot{a}(t)b(t) - a(t)\dot{b}(t)$. Clearly the conditions for positivity of the off-diagonal elements of $L(t)$ and $R(t)$ are not equivalent. Notice that if $S(t)$ is bistochastic, implying $a(t) = b(t)$, then also $L(t) = R(t)$.

Corollary 1. The classical dynamics is Schrödinger divisible if $\ell_k(t) \ge 0$ $(k = 1, 2)$. Similarly, it is Heisenberg divisible if $r_k(t) \ge 0$ $(k = 1, 2)$.

As an example consider two scenarios

$$a(t) = \frac{1+e^{-t}}{2}, \quad b(t) = \frac{1+e^{-2t}}{2} \tag{85}$$

and

$$a(t) = \frac{3+e^{-t}}{4}, \quad b(t) = \frac{1+3e^{-2t}\cos t}{4}. \tag{86}$$

The corresponding functions $\ell_k(t)$ and $r_k(t)$ are plotted in Fig. 3. For the first scenario, the dynamics is both Schrödinger and Heisenberg divisible. However, for the second one, it is Schrödinger divisible and Heisenberg indivisible. In particular for any $\mathbf{x} \in \mathbb{R}^2$, one has for both scenario

$$\frac{d}{dt}\|S(t)\mathbf{x}\|_1 \le 0, \tag{87}$$

while

$$\frac{d}{dt}\|S^\top(t)\mathbf{x}\|_\infty \le 0 \tag{88}$$

only holds for the first scenario, where $\|\mathbf{x}\|_\infty = \max_k |x_k|$ and $\|\mathbf{x}\|_1 = |x_1| + |x_2|$. Condition (88) is violated for Heisenberg indivisible dynamics.

Notice that, on the other hand, Heisenberg divisibility implies Schrödinger divisibility. In Appendix C, we show that $r_{1,2}(t) \ge 0$ implies $\ell_{1,2}(t) \ge 0$. Therefore, unlike the quantum case, for classical two-state systems, Heisenberg divisibility is stronger than Schrödinger divisibility.

IV. TIME EVOLUTION OF POVMs

In this section, we consider the connection between divisibility and two quantities widely used when studying POVMs: incompatibility and sharpness. Both quantities are contractive under CPU maps, and therefore nonmonotonicity is a signature of violations of Heisenberg divisibility.

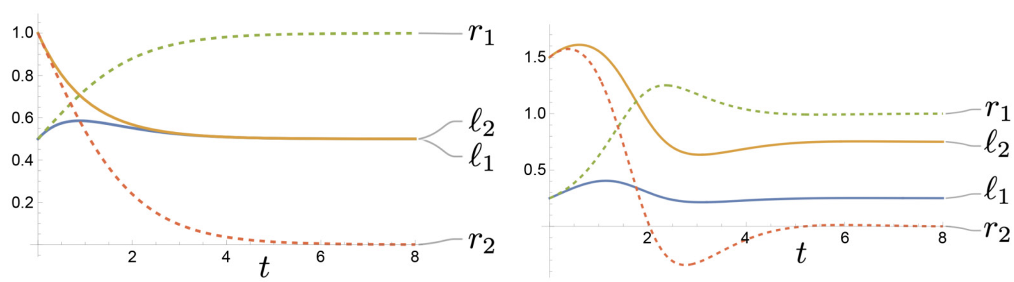

FIG. 3. Off-diagonal elements of $L(t)$ and $R(t)$. Solid: $\ell_{1,2}$ (81) and (82), dashed: $r_{1,2}$ (83) and (84). Left panel corresponds to (85), right panel to (86).

A. Compatibility

A quantum measurement can be described by a POVM (keeping for simplicity a discrete set of measurement outcomes) $M = (M_i)_i$, which assigns to any quantum state a probability distribution via $\rho \mapsto (\mathrm{tr}[\rho M_i])_i$. The elements of the POVM are effects $0 \le M_i \le \mathbb{1}$ satisfying $\sum_i M_i = \mathbb{1}$. Two POVMs $M = (M_i)_i$ and $N = (N_j)_j$ are said to be *compatible* or *jointly measurable* if there exists a third POVM $G = (G_{ij})_{i,j}$ such that [38]

$$M_i = \sum_j G_{ij}, \quad N_j = \sum_i G_{ij}. \tag{89}$$

If M and N are projective measurements, then compatibility is equivalent to commutativity $[M_i, N_j] = 0$, but in general commutativity is not necessary. Determining whether two POVMs are compatible is generally a hard task, but it drastically simplifies if one restricts to qubits [39].

The deviation from joint measurability can be quantified via *incompatibility monotones*: functions I on pairs of POVMs such that [40]

(1) $I(M, N) = 0$ if and only if M and N are compatible,
(2) $I(\Lambda[M], \Lambda[N]) \le I(M, N)$ for all CPU maps Λ.

In the context of Bell inequalities, incompatibility is a resource: two POVMs on Alice's side are incompatible if and only if there exist a pair of POVMs on Bob's side and an entangled bipartite state such that the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [41] is violated [42]. On the other hand, if all parties can perform the same set of measurements, any set of incompatible qubit measurements leads to the violation of some Bell inequalities in some suitable multiparticle scenario [43].

One possible incompatibility monotone can be defined starting from the noise-deformed POVM $M^{\lambda,p}$ with elements

$$M_i^{\lambda,p} := (1-\lambda)M_i + \lambda p_i \mathbb{1}, \tag{90}$$

which consists of mixing each POVM element with the identity, where $p = (p_i)_i$ is a probability distribution. It is then possible to define an incompatibility monotone as the minimal mixing needed for M and N to be compatible

$$I_p(M, N) := \inf\{\lambda \ge 0 \,|\, M^{\lambda,p}, N^{\lambda,p} \text{ are compatible}\}. \tag{91}$$

Another widely used incompatibility monotone is given by

$$\begin{aligned} I_{\mathrm{steer}}(M, N) := \inf\{\lambda \ge 0 \,|\, &(1-\lambda)M + \lambda\Lambda_0[M] \\ &\text{and } (1-\lambda)N + \lambda\Lambda_0[N] \text{ are compatible}\}, \end{aligned} \tag{92}$$

where $\Lambda_0[X] = \mathrm{tr}[X]\mathbb{1}/d$ is the completely depolarizing channel. It is possible to interpret $I_{\mathrm{steer}}(M, N)$ as a quantifier of incompatibility as a resource for steering [44,45]. In fact, it corresponds to the maximal noise that one can add to the maximally entangled state such that the resulting state is still steerable with measurements M and N [46].

Nonmonotonicity of incompatibility in time can be linked to violations of divisibility: because of contractivity of I under PU maps, if the dynamics is Heisenberg P-divisible then

$$\frac{d}{dt} I\left(\Phi_t^*(M), \Phi_t^*(N)\right) \le 0 \tag{93}$$

for all incompatibility monotones I and for all POVMs M and N. If revivals of I are witnessed, then the dynamics must necessarily violate Heisenberg divisibility. The dynamics of incompatibility in time was already studied in [47,48]; however, they considered commutative dynamics, for which Schrödinger and Heisenberg divisibility coincide. In Sec. V B, instead, we will provide explicit examples in which revivals of the resources quantified by I_p and I_{steer} are present even if the dynamics is Markovian in the Schrödinger picture. Similar results were derived in [49], in which it was shown that steering can be enhanced in noisy Schrödinger divisible qubit dynamics by applying a time-dependent control Hamiltonian. There, however, the revival of steering was not connected to violations of Heisenberg divisibility and, unlike in [47,48], the dynamics is not commutative due to the time-dependence of the Hamiltonian.

B. Sharpness

Another important quantity characterizing POVMs is that of *sharpness*, quantifying how close a POVM is to an ideal projective measurement. Similarly to incompatibility, sharpness can also be defined in nonunique ways. Given an effect E, a definition of the sharpness of the binary POVM $(E, \mathbb{1} - E)$ is [50]

$$\Sigma(E) := \|E\|_\infty + \|\mathbb{1} - E\|_\infty - 1. \tag{94}$$

Sharpness is contractive under PU maps, which follows from contractivity of the operator norm. Therefore, similarly to incompatibility, Heisenberg P-divisibility implies

$$\frac{d}{dt} \Sigma\left(\Phi_t^*[E]\right) \le 0. \tag{95}$$

Notice that this condition is not equivalent to contractivity of the operator norm (49) since it involves both E and $\mathbb{1} - E$. For instance, the phase covariant example of Eq. (59) presents revivals in the operator norm but not of sharpness: revivals of $\|E\|_\infty$ are compensated by decreases of $\|\mathbb{1} - E\|_\infty$. Nevertheless, in Sec. V B we will show an explicit example with a complete revival of sharpness for a dynamics Schrödinger divisible but Heisenberg indivisible. Again, it is possible to have revivals of resources in the Heisenberg picture without having non-Markovianity in the Schrödinger picture.

V. EXPLICIT CONDITIONS FOR QUBIT

In this section, we derive explicit conditions for P-divisibility for maps acting on qubits. The state of a qubit can be described via a three-dimensional vector $\mathbf{r} = (x, y, z)^\top$, $\|\mathbf{r}\| \le 1$ where $\|\cdot\|$ is the Euclidean norm, of the Bloch sphere $\mathcal{S}$ such that

$$\rho = \frac{1}{2}\left(\mathbb{1} + \mathbf{r}\cdot\sigma\right) \leftrightarrow \mathbf{r}, \tag{96}$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)^\top$. As a consequence, any CPTP map Φ can be written as a 4×4 matrix

$$\Phi \leftrightarrow M_\Phi = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{v} & \Lambda \end{pmatrix}, \tag{97}$$

where Λ is a 3×3 real matrix and $\mathbf{v}$ is a three-dimensional real vector, acting on the four-dimensional Bloch vector $\mathbf{r}_4 = (1, \mathbf{r})^\top$ so that the corresponding transformation of the Bloch sphere is the affine transformation

$$\Phi[\rho] \leftrightarrow \Lambda\mathbf{r} + \mathbf{v}. \tag{98}$$

The map Φ is positive if and only if it maps $\mathcal{S}$ into itself, while conditions for its complete positivity in terms of the Bloch representation have been derived in [51].

For self-adjoint operators, the Bloch representation consists of the four-dimensional real vector $\mathbf{a}_4 = (a_0, a_x, a_y, a_z)^\top = (a_0, \alpha)^\top$ such that

$$A = \frac{1}{2}\left(a_0\mathbb{1} + \alpha\cdot\sigma\right). \tag{99}$$

Furthermore, A is an effect if and only if [23]

$$0 \le a_0 \le 2, \quad \|\alpha\| \le \min\{a_0, 2 - a_0\}. \tag{100}$$

It is easy to show that

$$\mathrm{tr}[A\rho] = \mathbf{a}_4^\top \cdot \mathbf{r}_4 = a_0 + \alpha\cdot\mathbf{r}, \tag{101}$$

so that the matrix representation of the dual map Φ^* is given by

$$\begin{aligned} \mathrm{tr}[A\Phi[\rho]] &= \mathbf{a}_4^\top \cdot M_\Phi \mathbf{r}_4 = (M_\Phi^\top \mathbf{a}_4)^\top \cdot \mathbf{r}_4 \\ &= \mathrm{tr}[\Phi^*[A]\rho] = (M_{\Phi^*}\mathbf{a}_4)^\top \cdot \mathbf{r}_4, \end{aligned} \tag{102}$$

and therefore

$$M_{\Phi^*} = M_\Phi^\top. \tag{103}$$

Using (we omit time dependence for compactness)

$$M_{\dot\Phi} = \frac{d}{dt}M_\Phi, \quad M_{\Phi_1\Phi_2} = M_{\Phi_1}M_{\Phi_2}, \tag{104}$$

$$M_{\Phi^{-1}} = M_\Phi^{-1}, \tag{105}$$

it is straightforward to obtain the matrix representation of the left and right generators,

$$M_{\mathcal{L}} = \begin{pmatrix} 0 & \mathbf{0}^\top \\ \dot{\mathbf{v}} - \dot\Lambda\Lambda^{-1}\mathbf{v} & \dot\Lambda\Lambda^{-1} \end{pmatrix}, \tag{106}$$

$$M_{\mathcal{R}} = \begin{pmatrix} 0 & \mathbf{0}^\top \\ \Lambda^{-1}\dot{\mathbf{v}} & \Lambda^{-1}\dot\Lambda \end{pmatrix}, \tag{107}$$

and therefore the Heisenberg generators are

$$M_{\mathcal{L}^*} = \begin{pmatrix} 0 & (\dot{\mathbf{v}} - \dot\Lambda\Lambda^{-1}\mathbf{v})^\top \\ \mathbf{0} & \Lambda^{-\top}\dot\Lambda^\top \end{pmatrix}, \tag{108}$$

$$M_{\mathcal{R}^*} = \begin{pmatrix} 0 & (\Lambda^{-1}\dot{\mathbf{v}})^\top \\ \mathbf{0} & \dot\Lambda^\top\Lambda^{-\top} \end{pmatrix}, \tag{109}$$

where $\Lambda^{-\top} = (\Lambda^{-1})^\top = (\Lambda^\top)^{-1}$.

A. Analytic constraints on divisibility

In the Schrödinger picture, the dynamics is P-divisible if and only if $\Phi_{t+dt,t}$ maps the Bloch ball into itself. Using the Bloch matrix representation for $\Phi^L_{t+dt,t}$, one has

$$M_{\Phi^L_{t+dt,t}} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ dt\left(\dot{\mathbf{v}}_t - \dot\Lambda_t\Lambda_t^{-1}\mathbf{v}_t\right) & \mathbb{1} + dt\,\dot\Lambda_t\Lambda_t^{-1}, \end{pmatrix} \tag{110}$$

and therefore P-divisibility is equivalent to $M_{\Phi^L_{t+dt,t}}$ mapping $\mathcal{S}$ into itself

$$\left\|\left(\mathbb{1} + dt\,\dot\Lambda_t\Lambda_t^{-1}\right)\mathbf{r} + dt\left(\dot{\mathbf{v}}_t - \dot\Lambda_t\Lambda_t^{-1}\mathbf{v}_t\right)\right\| \le 1, \tag{111}$$

for all $\mathbf{r} \in \mathbf{S}$.

In the Heisenberg picture, P-divisibility corresponds to $\Phi^{R*}_{t+dt,t}$ being PU. Notice that positivity of $\Phi^{R*}_{t+dt,t}$ is equivalent to $\Phi^R_{t+dt,t}$ being PTP. On the Bloch sphere, the corresponding matrix reads

$$M_{\Phi^R_{t+dt,t}} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ dt\,\Lambda_t^{-1}\dot{\mathbf{v}}_t & \mathbb{1} + dt\,\Lambda_t^{-1}\dot\Lambda_t, \end{pmatrix} \tag{112}$$

and therefore Heisenberg P-divisibility corresponds to

$$\left\|\left(\mathbb{1} + dt\,\Lambda_t^{-1}\dot\Lambda_t\right)\mathbf{r} + dt\,\Lambda_t^{-1}\dot{\mathbf{v}}_t\right\| \le 1, \tag{113}$$

for all $\mathbf{r} \in \mathbf{S}$.

1. Unital maps

In the remaining, we will focus on unital maps in the Schrödinger picture (or, equivalently, on trace preserving maps in the Heisenberg picture). This condition is equivalent to $\mathbf{v}_t = \dot{\mathbf{v}}_t = 0$, and therefore the Bloch matrices for both $\Phi^L_{t+dt,t}$ and $\Phi^R_{t+dt,t}$ are in a block-diagonal form.

The condition for P-divisibility in the Schrödinger picture (111) takes the simpler form

$$\sup_{\mathbf{r}\in\mathcal{S}} \left\| \left(\mathbb{1} + dt\,\dot{\Lambda}_t\Lambda_t^{-1}\right)\mathbf{r}\right\|^2 = \left\|\mathbb{1} + dt\,\dot{\Lambda}_t\Lambda_t^{-1}\right\|^2_\infty \le 1, \tag{114}$$

while Heisenberg divisibility (113) reads

$$\left\|\mathbb{1} + dt\,\Lambda_t^{-1}\dot{\Lambda}_t\right\|^2_\infty \le 1. \tag{115}$$

These conditions can be simplified further by noticing that for an arbitrary matrix X, at first order in dt and assuming $\|\mathbf{r}\| = 1$, it holds

$$\|(\mathbb{1} + dt X)\,\mathbf{r}\|^2 = 1 + dt\,\langle \mathbf{r}, \left(X + X^\top\right)\mathbf{r}\rangle \tag{116}$$

and therefore $\|\mathbb{1} + dt X\|_\infty \le 1$ if and only if $X + X^\top \le 0$. Thus, Schrödinger divisibility is equivalent to [52]

$$X_S := \dot{\Lambda}_t\Lambda_t^{-1} + \Lambda_t^{-\top}\dot{\Lambda}^\top \le 0, \tag{117}$$

while for Heisenberg

$$X_H := \Lambda_t^{-1}\dot{\Lambda}_t + \dot{\Lambda}^\top\Lambda_t^{-\top} \le 0. \tag{118}$$

The map Λ can always be written as [51,53]

$$\Lambda = O_1 D O_2^\top, \tag{119}$$

where $D = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ and $O_{1,2}$ are orthogonal matrices with $\det O_{1,2} = 1$. To compute the P-divisibility conditions, we need to compute

$$\dot{\Lambda} = \dot{O}_1 D O_2^\top + O_1 \dot{D} O_2^\top + O_1 D \dot{O}_2^\top. \tag{120}$$

The derivative of the diagonal term is easily written as $\dot{D} = \mathrm{diag}\{\dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3\}$. The orthogonal terms, instead, can always be written as

$$O_i = e^{A_i}, \quad A_i^\top = -A_i. \tag{121}$$

Using the Baker-Campbell-Hausdorff formula, it is possible to compute $O_i(t+dt)$ at first order in dt as

$$\begin{aligned} e^{A_i(t+dt)} &= e^{A_i + dt\,\dot{A}_i} = e^{A_i} e^{dt\,\dot{A}_i} e^{C_i\,dt} \\ &= e^{A_i}\left(I + dt\,\dot{A}_i\right)(I + C_i\,dt) \\ &= e^{A_i}\left[I + dt\left(\dot{A}_i + C_i\right)\right] \\ &=: e^{A_i}[I + dt\,B_i], \end{aligned} \tag{122}$$

where C_i is an expression involving nested commutators of A_i and $\dot{A}_i$ and therefore $C_i^\top = -C_i$. Hence, it holds that

$$\dot{O}_i(t) = \frac{O_i(t+dt) - O_i(t)}{dt} = e^{A_i(t)}B_i(t), \tag{123}$$

with $B_i^\top(t) = -B_i(t)$. Therefore, using $\Lambda^{-1} = O_2 D^{-1} O_1^\top$ and $[D^{-1}, \dot{D}] = 0$, the conditions for P-divisibility in the Schrödinger and Heisenberg pictures of Eqs. (117) and (118) become

$$\begin{aligned} X_S = e^{A_1}\big[&2\dot{D}D^{-1} - DB_2D^{-1} \\ &+ D^{-1}B_2D\big]\,e^{-A_1} \le 0, \end{aligned} \tag{124}$$

$$\begin{aligned} X_H = e^{-A_2}\big[&2\dot{D}D^{-1} - D^{-1}B_1D \\ &+ DB_1D^{-1}\big]\,e^{A_2} \le 0. \end{aligned} \tag{125}$$

The first term $2\dot{D}D^{-1}$ is common to both divisibility conditions and its negativity is equivalent to monotonic contraction of the Bloch sphere $\dot{\lambda}_i \le 0$. The orthogonal transformations, instead, play a different and nontrivial role in the two pictures. In the Schrödinger picture, $O_1 = e^{A_1}$ does not play any role; O_2, on the other hand, can change the positivity of X_S whenever $\dot{O}_2 \neq 0$. The situation is analogous in the Heisenberg picture, upon swapping O_1 and O_2. Notice that we have considered unital maps only for the sake of simplicity: numerical evidence suggests that it is possible to find maps that are P-divisible only in one picture also when unitality is broken.

B. Example: dephasing and orthogonal transformation

We now proceed to derive explicit examples of dynamics either only divisible in the Schrödinger picture or only in the Heisenberg picture. Suppose that, at some given time $t > 0$, the diagonal term is such that $\dot{D} = 0$. Then the P-divisibility conditions of Eqs. (124) and (125) become

$$X_S = D^{-1}B_2D - DB_2D^{-1} \le 0, \tag{126}$$

$$X_H = DB_1D^{-1} - D^{-1}B_1D \le 0. \tag{127}$$

If $D = \lambda\mathbb{1}$ (or if $\dot{O}_1 = \dot{O}_2 = 0$), then both conditions are trivially satisfied, since $X_S = X_H = 0$. In the general case of nonisotropic shrinking and nonconstant orthogonal transformations, instead, such conditions are never satisfied: it always holds that $\mathrm{tr}X_S = \mathrm{tr}X_H = 0$, and therefore, if $X_{S,H} \neq 0$, they must have both positive and negative eigenvalues, thus violating divisibility. For instance, if $B_2 = 0$ ($\dot{O}_2 = 0$) and $B_1 \neq 0$ ($\dot{O}_1 \neq 0$), then the dynamics is not P-divisible in the Heisenberg picture, while it is just a unitary rotation in the Schrödinger picture, and therefore it

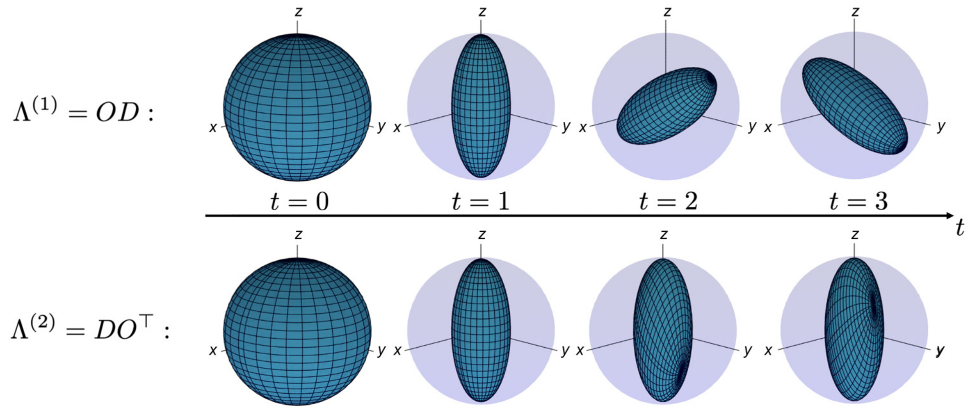

FIG. 4. Representation of the time evolution of the Bloch sphere under the maps $\Lambda^{(1,2)}$ of Eq. (128), in units $t_1 = 1$. For $t \leq 1$, $\Lambda^{(1)} = \Lambda^{(2)}$. At later times, $\Lambda^{(1)}$ is just an orthogonal rotation of the ellipsoid giving the evolution of the Bloch sphere, while $\Lambda^{(2)}$ does not change the image of the ellipsoid, with the image of the pure states moving along its surface.

is CP-divisible. If, instead, $B_2 \neq 0$ ($\dot{O}_2 \neq 0$) and $B_1 = 0$ ($\dot{O}_1 = 0$), it is the other way around.

Let us now consider a concrete example. More explicitly, we consider two dynamical maps $\Phi^{(1,2)}$ such that their Bloch matrix representations $\Lambda^{(1,2)}$ have only one of the orthogonal transformations nontrivial, namely

$$\Lambda^{(1)} = OD, \quad \Lambda^{(2)} = DO^{\top}. \tag{128}$$

In order to emphasize the different roles of D and O in the two pictures, we consider a scenario in which for $t < t_1$ only the diagonal term plays a role ($O(t < t_1) = \mathbb{1}$), while for $t \geq t_1$ only the orthogonal term does ($D(t \geq t_1) = D(t_1)$ or, equivalently, $\dot{D}(t \geq t_1) = 0$). For the sake of the example, we fix D as dephasing

$$D(t) = \mathrm{diag}\{\lambda(t), \lambda(t), 1\}, \tag{129}$$

with $\lambda(0) = 1$ and $\dot{\lambda}(t) \leq 0$ to ensure continuity and CP-divisibility in both pictures for $t < t_1$, and $\lambda(t \geq t_1) = \lambda(t_1)$ to have $\dot{D}(t \geq t_1) = 0$. The orthogonal transformation, instead, reads

$$O(t) = \exp[\beta(t)\Theta(t - t_1)L_x], \tag{130}$$

where Θ is the Heaviside theta function and L_x is the generator of the rotations in the x direction. A pictorial representation of the dynamics of the Bloch sphere under the maps $\Lambda^{(1,2)}$ of Eq. (128) is presented in Fig. 4, while the explicit form of $\lambda(t)$ and $\beta(t)$ is shown in the inset of Fig. 5. Animations representing the time evolution of the Bloch sphere can be found at [36].

For $t < t_1$, $\Lambda^{(1)}(t < t_1) = \Lambda^{(2)}(t < t_1)$ and the dynamics is CP-divisible in both pictures. For $t \geq t_1$, let us focus on $\Lambda^{(1)} = OD$. In the Schrödinger picture, it is straightforward to verify that $X_S^{(1)}(t \geq t_1) = 0$ and the dynamics is simply a rotation of the shrinked Bloch sphere

$$\mathcal{L}^{(1)}_{t \geq t_1}[\rho] = -i[H^{(1)}(t), \rho]. \tag{131}$$

In the Heisenberg picture, instead, the dynamics is not P-divisible, as it can be seen by explicitly computing the divisibility condition of Eq. (125), which reads

$$X_H^{(1)}(t \geq t_1) = \dot{\beta}(t)\left[D(t_1)L_x D^{-1}(t_1) - D(t_1)^{-1}L_x D(t_1)\right] \neq 0. \tag{132}$$

But, since $\mathrm{tr}X_H^{(1)}(t \geq t_1) = 0$, then necessarily $X_H^{(1)}(t \geq t_1) \not\leq 0$ and therefore the dynamics is not Heisenberg P-divisible for $t \geq t_1$. This fact can be seen by looking at the behavior in time of the trace distance D_1 and of the operator distance D_∞ (see the left panel of Fig. 5): the former is monotonically decreasing, corresponding to Schrödinger P-divisibility, while the latter is nonmonotonic, and therefore Heisenberg P-divisibility is violated. Therefore, it holds that $\mathcal{N}_S(\Phi^{(1)}) = 0$, while $\mathcal{N}_H(\Phi^{(1)*}) > 0$. In a similar manner, both incompatibility (91), (92) and sharpness (94) quantifiers are nonmonotonic (see the right panel of Fig. 5), showing that they witness Heisenberg divisibility violations.

On the other hand, if instead of $\Lambda^{(1)}$ we consider $\Lambda^{(2)} = DO^{\top}$, then the opposite scenario arises. In the Heisenberg picture, the dynamics is simply a unitary rotation and, correspondingly, $X_H^{(2)}(t \geq t_1) = 0$, implying P-divisibility. This time, instead, Schrödinger P-divisibility is violated, as it can be seen by computing

$$X_S^{(2)}(t \geq t_1) = -X_H^{(1)}(t \geq t_1) \not\leq 0. \tag{133}$$

Accordingly, it holds that $\mathcal{N}_S(\Phi^{(2)}) > 0$, while $\mathcal{N}_H(\Phi^{(2)*}) = 0$: now D_∞ is monotonic, while D_1 presents revivals. Because of Heisenberg divisibility, both incompatibility

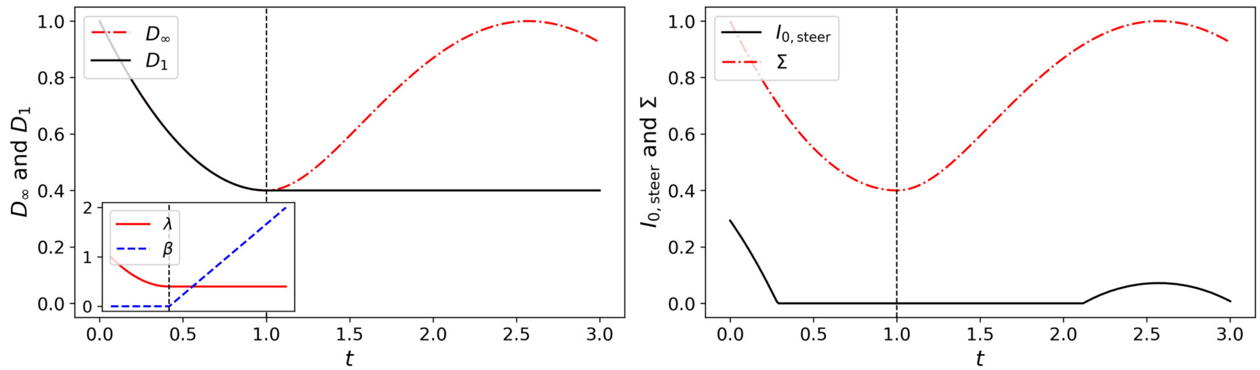

FIG. 5. Dynamics of Φ_t corresponding to $\Lambda^{(1)}$ of Eq. (128). Left panel: time evolution of $D_\infty\left(\Phi_t^*\left[|+_y\rangle\langle+_y|\right], \Phi_t^*\left[|-_y\rangle\langle-_y|\right]\right)$ and of $D_1\left(\Phi_t\left[|+_y\rangle\langle+_y|\right], \Phi_t\left[|-_y\rangle\langle-_y|\right]\right)$, where $|\pm_y\rangle$ are the eigenstates of σ_y. Since the dynamics is Schrödinger CP-divisible but Heisenberg P-indivisible, D_1 is monotonic while D_∞ is not. Inset: dynamical functions $\lambda(t)$ and $\beta(t)$ of Eqs. (129) and (130). Right panel: dynamics of incompatibility $I_0\left(\Phi_t^*\left[|+_y\rangle\langle+_y|\right], \Phi_t^*\left[|+_x\rangle\langle+_x|\right]\right)$ (for this particular dynamics, $I_0 = I_{\mathrm{steer}}$) and sharpness $\Sigma\left(\Phi_t^*\left[|+_y\rangle\langle+_y|\right]\right)$ of Eqs. (91), (92), and (94). $|+_x\rangle$ is the eigenstate of σ_x. Notice that the dynamics presents a full revival in sharpness and only a partial revival in incompatibility. The vertical lines represent the time at which Heisenberg P-divisibility is broken.

and sharpness are also monotonic. Notice that the dynamics is non-Markovian in the Schrödinger picture even if the image of the Bloch sphere remains constant: there is no expansion of the Bloch sphere, but non-Markovianity traces back to a nontrivial motion of its surface.

Therefore, considering situations for which $\dot{D} = 0$, it is possible to have dynamics that are not divisible either in the Schrödinger or in the Heisenberg picture by simply adding a nonconstant orthogonal transformation either to the right or to the left of D, respectively. By means of this simple example, we have shown that the two concepts of divisibility are not equivalent and it can be violated in either only one picture or in both at the same time.

Moreover, the example allows one to highlight the crucial role of the orthogonal transformations in determining nondivisibility. Indeed, if only D is present ($O(t) = \mathbb{1}$ also for $t \geq t_1$), then $\Lambda^{(1)} = \Lambda^{(2)} = D$ and the dynamics is divisible in both pictures. The addition of a nontrivial orthogonal transformation O either to the left or to the right of D allows the dynamics in one picture to be non-Markovian, although corresponding only to a unitary rotation in the other picture. This fact is possible because O breaks the commutativity (11). Therefore, even if the condition of Schrödinger P-divisibility (33) does not depend on the Hamiltonian part of the generator, such nontrivial Hamiltonian in the Schrödinger picture might alter the divisibility in the Heisenberg picture.

Notice that in this example the dynamics is not smooth at $t = t_1$; however, the definition of O and D can be changed to have smooth versions of the Θ function and the behavior in time is left unchanged. Similarly, in order to emphasize the different roles of the diagonal and of the orthogonal parts of the dynamics, we have considered the simplified scenario in which only either D or O is nonconstant at each time. However, if we omit the theta function Θ altogether in the definition of O in Eq. (130), the qualitative behavior of the two time evolutions is the same: $\Lambda^{(1)}$ presents violations of P-divisibility only in the Heisenberg picture, while $\Lambda^{(2)}$ only in the Schrödinger picture.

VI. CONCLUSIONS

In this work, we have introduced the concept of divisibility for dynamical evolutions in the Heisenberg picture. While the Schrödinger and Heisenberg pictures are indeed physically equivalent in terms of observable predictions, we have shown that divisibility in one picture does not imply divisibility in the other. This happens because the time-ordered Heisenberg picture propagator $\Phi_{t,s}^{R*}$ is not generated by the dual of the Schrödinger generator, and therefore its (complete) positivity is not equivalent to the (complete) positivity of the Schrödinger propagator $\Phi_{t,s}^{L}$. We have illustrated this fact with concrete examples whose evolution is divisible only in one picture. This inequivalence implies that common quantifiers of non-Markovianity may miss relevant features of the dynamics if it is considered only in the Schrödinger picture.

We have also provided an operational interpretation for violations of P-divisibility in the Heisenberg picture which is dual to the interpretation in the Schrödinger picture: just like violations in the Schrödinger picture imply a nonmonotonic probability of guessing which of two states is prepared, violations in the Heisenberg picture imply nonmonotonicity in the guessing probability of which of two effects is measured. According to this interpretation, we have also provided a measure of the total violation of divisibility in the Heisenberg picture analogous to the measure of non-Markovianity in the Schrödinger picture. Such violations of divisibility can also be connected to a nonmonotonic behavior of compatibility or sharpness of POVMs.

Our work shows that memory effects can be present in the dynamics even without corresponding to violations of divisibility in the Schrödinger picture, since divisibility can be violated only in the Heisenberg picture. These memory effects can be seen from the nonmonotonic guessing probability between effects. Our results align with the findings of [54], in which it was shown that CP divisibility in the Schrödinger picture does not imply Markovianity. This also suggests that CP divisibility, either in the Schrödinger or in the Heisenberg picture, and quantum Markovianity, as characterized in terms of process tensors [55,56], can be two physically and operationally distinct concepts.

Future work will be devoted to further explore the intricate connection between divisibility in the two pictures. We aim to derive counterparts of other quantifiers of non-Markovianity in the Schrödinger picture to the Heisenberg picture, both operational and nonoperational ones [57] connect it to the presence of classical or quantum memory [58–60], and causal or noncausal revival of information [61], as well as exploring whether Heisenberg divisibility is equivalent to monotonic decrease of information [5]. We also aim to connecting Heisenberg indivisibility to concrete enhancement in tasks hindered by noise.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [36].

APPENDIX A: PROOF OF Eq. (45)

Consider two effects $0 \leq E, F \leq \mathbb{1}$ and a state ρ. Each effect gives rise to a two-valued probability distribution (the probability of obtaining a "yes" or "no" outcome to the associated measurement) via

$$p_E := (p_E(y), p_E(n)) = (\mathrm{tr}[E\rho], 1 - \mathrm{tr}[E\,\rho]) \tag{A1}$$

and similarly for F. The probability of committing an error when guessing from which probability distribution the observation is from is given by [22]

$$P_{\mathrm{err}}(p_E, p_F|\rho) = \frac{1}{2}\sum_{x\in y,n} \min\{p_E(x), p_F(x)\}. \tag{A2}$$

Suppose, without loss of generality, that $p_F(y) \leq p_E(y)$, which implies $p_F(n) \geq p_E(n)$, then

$$\begin{aligned} P_{\mathrm{err}}(p_E, p_F|\rho) &= \frac{1}{2}\left(p_F(y) + p_E(n)\right) \\ &= \frac{1}{2} - \frac{1}{2}\mathrm{tr}[(E-F)\rho]. \end{aligned} \tag{A3}$$

The other case $p_F(y) \geq p_E(y)$ just corresponds to swapping E and F, and therefore

$$P_{\mathrm{err}}(p_E, p_F|\rho) = \frac{1}{2} - \frac{1}{2}\left|\mathrm{tr}\left[(E-F)\rho\right]\right|. \tag{A4}$$

Using the optimal strategy corresponds to minimizing the error probability, giving

$$\begin{aligned} P_{\mathrm{err}}(E,F) &= \inf_\rho P_{\mathrm{err}}(p_E, p_F|\rho) \\ &= \frac{1}{2} - \frac{1}{2}\sup_\rho \left|\mathrm{tr}\left[(E-F)\rho\right]\right| \\ &= \frac{1}{2} - \frac{1}{2}\sup_\rho \mathrm{tr}\left[|E-F|\,\rho\right] \\ &= \frac{1}{2} - \frac{1}{2}D_\infty(E,F). \end{aligned} \tag{A5}$$

Thus, the probability of correctly guessing whether E or F was measured is

$$\begin{aligned} P^{\mathrm{e}}_{\mathrm{guess}}(E,F) &= 1 - P_{\mathrm{err}}(E,F) \\ &= \frac{1}{2}\left(1 + D_\infty(E,F)\right). \end{aligned} \tag{A6}$$

APPENDIX B: PROOF OF Eq. (50)

The proof of the information bound for D_1 used in [24,62,63] applies to any distance $D(X,Y) = \|X-Y\|$ provided that the norm $\|\cdot\|$ is contracting under the action of completely positive trace preserving transformations and can be further extended to entropic distinguishability quantifiers [9,10]. For the operator distance D_∞ of Eq. (46), the contractivity property does not hold in general and in particular when the considered completely positive trace preserving transformation is the partial trace we rather have [64,65]

$$\|\mathrm{tr}_E A_{SE}\|_\infty \leq d_E \|A_S\|_\infty, \tag{B1}$$

where d_E is the dimension of the Hilbert space of the environment. The relevant bound on the variation of the distinguishability quantifier D_∞ has to be obtained working in the Heisenberg picture, therefore with the dual maps. We first recall that the dual map to the partial trace for the states is given by the completely positive unital transformation [18]

$$(\mathrm{tr}_E)^*[X_S] = X_S \otimes \mathbb{1}_E, \tag{B2}$$

so that in correspondence to the two distinct effects X_S^1 and X_S^2 it is natural to consider the pair of system-environment observables given by $X_S^1 \otimes \mathbb{1}_E$ and $X_S^2 \otimes \mathbb{1}_E$. On the other hand, the dual map to the assignment map

$$\mathcal{A}_{\rho_E}[\rho_S] = \rho_S \otimes \rho_E, \tag{B3}$$

that is used to determine the initial system-environment state given an initial system state, is given by [18]

$$\mathcal{A}^*_{\rho_E}[X_{SE}] = \mathrm{tr}_E\{\rho_E X_{SE}\}, \tag{B4}$$

providing the relevant connection between a system-environment observable and the corresponding observable for observations on the system only, obtained averaging out the environmental degrees of freedom. We now define as usual the unitarily evolved system-environment operators as

$$X_{SE}(t) = U_{SE}(t)^\dagger X_S \otimes \mathbb{1}_E U_{SE}(t), \tag{B5}$$

where $U_{SE}(t)$ is the overall unitary evolution operator and the relevant observables are chosen according to Eq. (B2). We further introduce

$$\begin{aligned} X_S(t) &= \mathcal{A}^*_{\rho_E}[X_{SE}(t)] \\ &= \mathrm{tr}_E\{\rho_E U_{SE}(t)^\dagger X_S \otimes \mathbb{1}_E U_{SE}(t)\} \end{aligned} \tag{B6}$$

and correspondingly

$$\begin{aligned} X_E(t) &= \mathcal{A}^*_{\rho_S}[X_{SE}(t)] \\ &= \mathrm{tr}_S\{\rho_S U_{SE}(t)^\dagger X_S \otimes \mathbb{1}_E U_{SE}(t)\}, \end{aligned} \tag{B7}$$

according to Eq. (B4). We now exploit contractivity of the uniform norm with respect to positive unital transformations, so that we have

$$\begin{aligned} \|X_S^1(t) - X_S^2(t)\|_\infty &= \|\mathcal{A}^*_{\rho_E}[X_{SE}^1(t)] - \mathcal{A}^*_{\rho_E}[X_{SE}^2(t)]\|_\infty \\ &\leq \|X_{SE}^1(t) - X_{SE}^2(t)\|_\infty, \end{aligned} \tag{B8}$$

and further using invariance under unitary transformations of the uniform norm

$$\|X_S^1(t) - X_S^2(t)\|_\infty \leqslant \|X_{SE}^1(s) - X_{SE}^2(s)\|_\infty. \tag{B9}$$

We then exploit the identity

$$\begin{aligned} X_{SE}^1(s) - X_{SE}^2(s) = X_{SE}^1(s) &- X_S^1(s) \otimes X_E^1(s) \\ &+ X_S^1(s) \otimes X_E^1(s) - X_S^1(s) \otimes X_E^2(s) \\ &+ X_S^1(s) \otimes X_E^2(s) - X_S^2(s) \otimes X_E^2(s) \\ &+ X_S^2(s) \otimes X_E^2(s) - X_{SE}^2(s) \end{aligned} \tag{B10}$$

that combined with the triangle inequality and the following property of the uniform norm

$$\|AB\|_\infty \leqslant \|A\|_\infty \|B\|_\infty \tag{B11}$$

leads to

$$\begin{aligned} \|X_S^1(t) - X_S^2(t)\|_\infty \leq{}& \|X_{SE}^1(s) - X_S^1(s) \otimes X_E^1(s)\|_\infty \\ &+ \|X_E^1(s) - X_E^2(s)\|_\infty \|X_S^1(s)\|_\infty \\ &+ \|X_S^1(s) - X_S^2(s)\|_\infty \|X_E^2(s)\|_\infty \\ &+ \|X_S^2(s) \otimes X_E^2(s) - X_{SE}^2(s)\|_\infty. \end{aligned} \tag{B12}$$

For any effect X_S we furthermore have exploiting contractivity of the uniform norm with respect to positive unital transformations together with invariance under unitary transformations

$$\begin{aligned} \|X_S(s)\|_\infty &= \|\mathcal{A}^*_{\rho_S}[X_{SE}(t)]\|_\infty \\ &\leq \|X_{SE}(t)\|_\infty \\ &= \|X_S \otimes \mathbb{1}_E\|_\infty \\ &= \|X_S\|_\infty \\ &\leq 1. \end{aligned} \tag{B13}$$

Combining these inequalities we finally obtain

$$\begin{aligned} \|X_S^1(t) - X_S^2(t)\|_\infty - \|X_S^1(s) - X_S^2(s)\|_\infty \leq{}& \|X_E^1(s) \\ &- X_E^2(s)\|_\infty + \|X_{SE}^1(s) - X_S^1(s) \otimes X_E^1(s)\|_\infty \\ &+ \|X_{SE}^2(s) - X_S^2(s) \otimes X_E^2(s)\|_\infty \end{aligned} \tag{B14}$$

and therefore Eq. (50).

APPENDIX C: CONNECTION BETWEEN HEISENBERG AND SCHRÖDINGER DIVISIBILITY FOR THE CLASSICAL TWO-STATE SYSTEM

We now show that positivity of $r_{1,2}(t)$ of Eqs. (83) and (84) implies the positivity of $\ell_{1,2}(t)$ of Eqs. (81) and (82). Consider first the case in which the denominator is positive, i.e., $a(t) + b(t) - 1 \geq 0$. Positivity of $r_{1,2}(t)$ is equivalent to $\dot{a}(t) \leq 0$ and $\dot{b}(t) \leq 0$. Using the fact that

$a(t) - 1 \leq 0$, it is easy to show that the numerator of $\ell_1(t)$ is also positive,

$$-w(t) - \dot{b}(t) = \dot{b}(t)\,[a(t) - 1] - \dot{a}(t)b(t) \geq 0, \qquad \text{(C1)}$$

since both terms are positive. Similarly, using $b(t) - 1 \leq 0$, also the numerator of $\ell_2(t)$ is positive,

$$w(t) - \dot{a}(t) = \dot{a}(t)\,[b(t) - 1] - \dot{b}(t)a(t) \geq 0, \qquad \text{(C2)}$$

and therefore $\ell_{1,2}(t) \geq 0$.

On the other hand, if $a(t) + b(t) - 1 < 0$ Heisenberg divisibility is equivalent to $\dot{a}(t) \geq 0$ and $\dot{b}(t) \geq 0$, and in a similar manner it is easy to show that the numerators of $\ell_{1,2}$ are now negative

$$\dot{b}(t)\,[a(t) - 1] - \dot{a}(t)b(t) \leq 0, \qquad \text{(C3)}$$

$$\dot{a}(t)\,[b(t) - 1] - \dot{b}(t)a(t) \leq 0, \qquad \text{(C4)}$$

but, since the denominator is also negative, also in this case $\ell_{1,2}(t) \geq 0$.

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