

# Quantum optical versus quantum Brownian motion master equation in terms of covariance and equilibrium properties

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Structures of quantum Fokker–Planck equations are characterized with respect to the properties of complete positivity, covariance under symmetry transformations and satisfaction of equipartition, referring to recent mathematical work on structures of unbounded generators of covariant quantum dynamical semigroups. In particular the quantum optical master equation and the quantum Brownian motion master equation are shown to be associated to  $U(1)$  and  $R$  symmetry, respectively. Considering the motion of a Brownian particle, where the expression of the quantum Fokker–Planck equation is not completely fixed by the aforementioned requirements, a recently introduced microphysical kinetic model is briefly recalled, where a quantum generalization of the linear Boltzmann equation in the small energy and momentum transfer limit straightforwardly leads to quantum Brownian motion. © 2002 American Institute of Physics. [DOI: 10.1063/1.1505126]

## I. INTRODUCTION

A theory of quantum dissipation, even restricted to the Markovian case, is a subject of major interest for many different scientific communities, ranging from mathematicians to physicists and chemists, according to various perspectives. Among these in first line experimental applications in phenomena where spontaneous emission, decoherence and dissipation play an important role, but also theoretical studies regarding the connection between quantum and classical description of dynamics, since, thanks to the lack of simple quantization recipes, such as the correspondence principle, dissipative systems become a fruitful working area where typical quantum structures may emerge. This interest has led to a huge number of proposals of Markovian master equations for the description of such dissipative phenomena, based on microphysical, phenomenological or purely mathematical approaches (see references in Refs. 1–3), not always accompanied by clear statements with regard to obeyed physical and mathematical properties, thus often leading to amendments of these models in view of some missing desired feature. This is in particular true with respect to the property of positivity or complete positivity,<sup>4</sup> proper distinction between Hamiltonian and dissipative part,<sup>5</sup> translational invariance,<sup>6,7</sup> and decoherence effects.<sup>8</sup>

As a result some efforts have been made to compare and characterize the different proposals in view of relevant mathematical and physical properties: preservation of the positivity of the statistical operator, existence of a suitable canonical stationary state, and translational invariance.<sup>9–11,12</sup> The main starting point for this research work was the result of Lindblad for the most general structure of bounded generator of a completely positive quantum dynamical semigroup,<sup>13</sup> together with his paper on quantum Brownian motion,<sup>14</sup> in which some of these issues were already considered. Complete positivity, actually equivalent to positivity for the considered Markovian quasi-free systems,<sup>15</sup> ensures that the statistical operator preserves positivity during the time evolution, and has emerged as a typical quantum feature, corresponding to the requirement that positivity of the time evolution is preserved under entanglement. It is in fact by

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now an essential property in the realm of quantum communication,<sup>16</sup> even though its origins lie in the theory of quantum measurement.<sup>17</sup> The result of Lindblad rigorously holds for bounded operators, though it is usually exploited as a starting point also for unbounded operators, leaving in this case the task open, to show that the considered structure is a proper generator of a completely positive quantum dynamical semigroup.

In this article we will try to further clarify the situation, showing that if besides complete positivity and equipartition, i.e., existence of a suitable canonical stationary solution, proper covariance properties of the generator of the dynamics, reflecting the relevant symmetries of the reservoir (and therefore not necessarily only translational invariance), are taken into account, a suitable characterization of two different classes of master equations can be given in a very neat way. In particular, with the two one-dimensional Lie groups  $U(1)$  and  $R$  two distinct type of master equations are associated, describing respectively the damped harmonic oscillator (the so-called quantum optical master equation) and the motion of a Brownian particle (the so-called quantum Brownian motion master equation), which, despite the fact that their physical realm of validity is essentially well-understood,<sup>18</sup> are sometimes mixed up,<sup>7</sup> their characterization in connection to underlying symmetry of the reservoir being usually neglected.<sup>18</sup> Recent work on completely positive quantum dynamical semigroups has shown that, asking for suitable covariance properties, a characterization of generators of such semigroups can be given also in the case of unbounded operators, where very few results are available,<sup>19</sup> so that one can check whether a proposed formal Lindblad structure is indeed a proper generator of a quantum dynamical semigroup. The room left by these mathematical and physical requirements should be covered by microphysical approaches, determining their relevance and predictive power. It turns out that the quantum optical master equation is in essence fixed by these requirements, while in the quantum Brownian motion case there is a nontrivial freedom left, thus explaining the huge, sometimes contradictory, literature devoted to the quantum Brownian motion master equation. In this connection a recently obtained<sup>20–22</sup> microphysical model for the quantum description of the motion of a Brownian particle is presented, derived from a quantum version of the linear Boltzmann equation,<sup>23,24</sup> extending previous phenomenological models<sup>25</sup> where dissipation effects leading to the correct stationary solution could not be accounted for. For further work relying on symmetry properties in the case of a fermionic oscillator, see Ref. 26.

The article is organized as follows: in Sec. II we recall the most general Lindblad structure corresponding to a quantum Fokker–Planck equation, further showing the expressions that come out asking for shift-covariance or translation-covariance; in Sec. III we outline the results of a microphysical model for the description of quantum Brownian motion, obtained from a quantum linear Boltzmann equation expressed in terms of the operator-valued dynamic structure factor of the reservoir; in Sec. IV we briefly comment on our results, indicating possible future extensions.

## II. GENERAL EXPRESSION OF THE QUANTUM FOKKER–PLANCK EQUATION WITH A LINDBLAD STRUCTURE

In order to give a quantum description of dissipative phenomena which at the classical level are described by second-order Fokker–Planck equations, we first concentrate on generalizations of the quantum Liouville equation which preserve trace and positivity of the statistical operator, also possibly accounting for friction effects. This can be done considering the most general expression of generator of a completely positive quantum dynamical semigroup (complete positivity being for these Markovian quasi-free systems necessary in order to preserve positivity<sup>15</sup>), given by a Lindblad structure in which position and momentum operator for the microsystem (respectively  $\hat{x}$  and  $\hat{p}$ ) appear restricted to bilinear expressions, according to the classical approximations leading to a friction term proportional to velocity. The general expression of a Markovian quantum Fokker–Planck equation preserving the positivity of the statistical operator is thus given, in the one-dimensional case to which we will restrict for simplicity, by<sup>14,27,10</sup>

$$\mathcal{M}_{\hat{x}\hat{p}}[\hat{\rho}] = -\frac{i}{\hbar} \left[ H_0(\hat{x}, \hat{p}) + \frac{\mu}{2} \{\hat{x}, \hat{p}\}, \hat{\rho} \right] + \sum_{i=1}^2 \left[ \hat{V}_i \hat{\rho} \hat{V}_i^\dagger - \frac{1}{2} \{\hat{V}_i^\dagger \hat{V}_i, \hat{\rho}\} \right], \tag{1}$$

$$\hat{V}_i = \alpha_i \hat{p} + \beta_i \hat{x}, \quad \alpha_i, \beta_i \in \mathbb{C}, \quad \mu \in \mathbb{R},$$

where  $H_0$  is a self-adjoint operator given by a quadratic expression in  $\hat{x}$  and  $\hat{p}$  describing the free system, and the added Hamiltonian term proportional to  $\mu$  has been introduced for later convenience. For the sake of comparison with classical Fokker–Planck equations, in order to make the intuitive physical meaning of the different contributions clear, (1) is usually conveniently written in the following form using nested commutators and anticommutators:<sup>10</sup>

$$\begin{aligned} \mathcal{L}_{\hat{x}\hat{p}}[\hat{\rho}] = & -\frac{i}{\hbar} [H_0(\hat{x}, \hat{p}), \hat{\rho}] - \frac{i(\mu - \gamma)}{\hbar} [\{\hat{x}, \hat{p}\}, \hat{\rho}] - \frac{i}{\hbar} \gamma [\hat{x}, \{\hat{p}, \hat{\rho}\}] - \frac{D_{pp}}{\hbar^2} [\hat{x}, [\hat{x}, \hat{\rho}]] \\ & - \frac{D_{xx}}{\hbar^2} [\hat{p}, [\hat{p}, \hat{\rho}]] + \frac{D_{px}}{\hbar^2} [\hat{x}, [\hat{p}, \hat{\rho}]] + \frac{D_{xp}}{\hbar^2} [\hat{p}, [\hat{x}, \hat{\rho}]], \end{aligned} \tag{2}$$

where due to  $[\hat{p}, [\hat{x}, \cdot]] = [\hat{x}, [\hat{p}, \cdot]]$  actually  $D_{xp} = D_{px}$  and the new coefficients are related to  $\alpha_i$  and  $\beta_i$  through the equations

$$D_{xx} = \frac{\hbar}{2} \sum_{i=1}^2 |\alpha_i|^2, \quad D_{pp} = \frac{\hbar}{2} \sum_{i=1}^2 |\beta_i|^2, \quad D_{px} = -\frac{\hbar}{2} \Re \sum_{i=1}^2 \alpha_i^* \beta_i, \quad \gamma = \frac{\hbar}{2} \Im \sum_{i=1}^2 \alpha_i^* \beta_i,$$

so that the following inequalities hold,

$$D_{xx} \geq 0, \quad D_{pp} \geq 0, \quad D_{xx} D_{pp} - D_{px}^2 \geq \frac{\gamma^2 \hbar^2}{4}, \tag{3}$$

which are necessary and sufficient conditions for an expression of the form (2) to be cast in Lindblad form, corresponding to the requirement that the matrix of coefficients<sup>27</sup>

$$\mathbf{D} = \begin{pmatrix} D_{xx} & D_{px} + i \frac{\hbar}{2} \gamma \\ D_{px} - i \frac{\hbar}{2} \gamma & D_{pp} \end{pmatrix} \tag{4}$$

has a non-negative determinant.

An alternative but equivalent expression for (2) can be given introducing, with the aid of a length  $l$ , whose physical meaning and expression will depend on the system to be described, creation and annihilation operators  $\hat{a}$  and  $\hat{a}^\dagger$ :

$$\hat{a} = \frac{1}{l\sqrt{2}} \left( \hat{x} + \frac{i}{\hbar} l^2 \hat{p} \right) \quad \hat{a}^\dagger = \frac{1}{l\sqrt{2}} \left( \hat{x} - \frac{i}{\hbar} l^2 \hat{p} \right), \tag{5}$$

satisfying the commutation relation  $[\hat{a}, \hat{a}^\dagger] = 1$ . One thus obtains the expression

$$\begin{aligned} \mathcal{L}_{\hat{a}\hat{a}^\dagger}[\hat{\rho}] = & -\frac{i}{\hbar} [H_0(\hat{a}, \hat{a}^\dagger), \hat{\rho}] - \frac{(\mu - \gamma)}{2} [\hat{a}^2 - \hat{a}^{\dagger 2}, \hat{\rho}] \\ & - \frac{\gamma}{2} [[\hat{a}, \{\hat{a}, \hat{\rho}\}] - [\hat{a}, \{\hat{a}^\dagger, \hat{\rho}\}]] + \text{h.c.} \\ & - \frac{1}{2} \left( \frac{D_{xx}}{l^2} + \frac{D_{pp} l^2}{\hbar^2} \right) [\hat{a}^\dagger, [\hat{a}, \hat{\rho}]] + \frac{1}{2} \left( \frac{D_{xx}}{l^2} - \frac{D_{pp} l^2}{\hbar^2} - 2i \frac{D_{px}}{\hbar} \right) [\hat{a}, [\hat{a}, \hat{\rho}]] + \text{h.c.}, \end{aligned}$$

or, equivalently, collecting terms as in (1),

$$\begin{aligned}
 \mathcal{M}_{\hat{a}\hat{a}^\dagger}[\hat{\rho}] = & -\frac{i}{\hbar} [H_0(\hat{a}, \hat{a}^\dagger), \hat{\rho}] - \frac{\mu}{2} [\hat{a}^2 - \hat{a}^{\dagger 2}, \hat{\rho}] \\
 & + \left( \frac{D_{xx}}{l^2} + \frac{D_{pp}l^2}{\hbar^2} + \gamma \right) \left[ \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2} \{\hat{\rho}, \hat{a}^\dagger\hat{a}\} \right] \\
 & + \left( \frac{D_{xx}}{l^2} + \frac{D_{pp}l^2}{\hbar^2} - \gamma \right) \left[ \hat{a}^\dagger\hat{\rho}\hat{a} - \frac{1}{2} \{\hat{\rho}, \hat{a}\hat{a}^\dagger\} \right] \\
 & - \left( \frac{D_{xx}}{l^2} - \frac{D_{pp}l^2}{\hbar^2} - 2i\frac{D_{px}}{\hbar} \right) \left[ \hat{a}\hat{\rho}\hat{a} - \frac{1}{2} \{\hat{\rho}, \hat{a}^2\} \right] + \text{h.c.} \tag{6}
 \end{aligned}$$

The recalled expressions essentially give the possible Lindblad structures at most bilinear in the operators  $\hat{x}$  and  $\hat{p}$  or  $\hat{a}$  and  $\hat{a}^\dagger$ .

**A. Shift-covariance**

We now analyze the behavior of the considered expressions with respect to suitable symmetry transformations. Consider a locally compact group  $G$  and a unitary representation  $\hat{U}(g)$ , with  $g \in G$ , on the Hilbert space of the system: following Ref. 28 we say that a mapping  $\mathcal{F}$  in the Schrödinger picture is  $G$ -covariant if it commutes with the mapping  $\mathcal{U}_g[\cdot] = \hat{U}(g) \cdot \hat{U}^\dagger(g)$  for all  $g \in G$ :

$$\mathcal{F}[\mathcal{U}_g[\cdot]] = \mathcal{U}_g[\mathcal{F}[\cdot]]. \tag{7}$$

Let us now consider the following unitary representation of the group  $U(1)$ ,

$$\hat{U}_\phi = e^{i\phi\hat{N}},$$

where  $\hat{N} = \hat{a}^\dagger\hat{a}$  is the number operator and  $\phi \in [0, 2\pi]$ . If we now ask for the general expression (6) invariance under the action of the group  $U(1)$ , i.e., shift-covariance according to<sup>29</sup>

$$\mathcal{M}_{\hat{a}\hat{a}^\dagger}[e^{i\phi\hat{N}} \cdot e^{-i\phi\hat{N}}] = e^{i\phi\hat{N}} \mathcal{M}_{\hat{a}\hat{a}^\dagger}[\cdot] e^{-i\phi\hat{N}}, \tag{8}$$

then the Hamiltonian has to be a function of the generator of the transformation  $\hat{N}$ , and the following stringent requirements appear:

$$D_{xx} = D_{pp} \frac{l^4}{\hbar^2}, \quad D_{px} = 0, \quad \mu = 0.$$

Note that the condition  $\mu = 0$  appears here as a necessary condition for shift-covariance or invariance under the relevant symmetry group and not as a natural or most simple choice as sometimes advocated.<sup>2</sup> From a physical point of view the master equation is expected to reflect the relevant symmetry of the reservoir the microscopic system is interacting with. Considering, for example, a single mode (harmonic oscillator) interacting with the electro-magnetic field, one has a  $U(1)$  symmetry and condition (8) is actually equivalent to the rotating wave approximation, essentially saying that the master equation is invariant under the transformation

$$\hat{a} \rightarrow \hat{a}e^{-i\theta}, \quad \hat{a}^\dagger \rightarrow \hat{a}^\dagger e^{+i\theta},$$

or, equivalently, in terms of  $\hat{x}$  and  $\hat{p}$ ,

$$\hat{x} \rightarrow \hat{x} \cos \theta + \hat{p} \sin \theta, \quad \hat{p} \rightarrow -\hat{x} \sin \theta + \hat{p} \cos \theta. \tag{9}$$

The rotating wave approximation is therefore strictly linked to a  $U(1)$  symmetry and one cannot expect or try to obtain translational invariance in this case.<sup>7</sup>

If now one further makes the choice  $H_0(\hat{N}) = \hbar \omega (\hat{N} + \frac{1}{2})$  corresponding to a single mode (harmonic oscillator) and asks that an operator with the canonical structure  $\hat{\rho}_0 = e^{-\beta H_0(\hat{N})}$  be a stationary solution, i.e.,  $\mathcal{M}_{\hat{a}\hat{a}^\dagger}[\hat{\rho}_0] = 0$ , a further connection between the coefficients of the master equation appears,<sup>10</sup>

$$2 \frac{D_{pp} l^2}{\hbar^2} = \gamma \coth\left(\frac{1}{2} \beta \hbar \omega\right),$$

where  $\beta$  is the inverse temperature characterizing the thermal electromagnetic field the mode is interacting with. The requirements of complete positivity, shift-covariance and existence of the expected canonical stationary solution then fix the quantum Fokker–Planck equation to be of the form

$$\begin{aligned} \mathcal{M}_{\hat{a}\hat{a}^\dagger}^{\text{QO}}[\hat{\rho}] = & -\frac{i}{\hbar} [H_0(\hat{N}), \hat{\rho}] \\ & + \gamma \left[ \coth\left(\frac{1}{2} \beta \hbar \omega\right) + 1 \right] \left[ \hat{a} \hat{\rho} \hat{a}^\dagger - \frac{1}{2} \{\hat{\rho}, \hat{a}^\dagger \hat{a}\} \right] \\ & + \gamma \left[ \coth\left(\frac{1}{2} \beta \hbar \omega\right) - 1 \right] \left[ \hat{a}^\dagger \hat{\rho} \hat{a} - \frac{1}{2} \{\hat{\rho}, \hat{a} \hat{a}^\dagger\} \right], \end{aligned} \tag{10}$$

or in terms of the average of the number operator over a thermal distribution,

$$N_\beta(\omega) = \frac{1}{e^{\beta \hbar \omega} - 1} = \frac{1}{2} \left[ \coth\left(\frac{1}{2} \beta \hbar \omega\right) - 1 \right],$$

setting  $\eta = 2 \gamma$ ,

$$\mathcal{M}_{\hat{a}\hat{a}^\dagger}^{\text{QO}}[\hat{\rho}] = -\frac{i}{\hbar} [H_0(\hat{N}), \hat{\rho}] + \eta (N_\beta(\omega) + 1) \left[ \hat{a} \hat{\rho} \hat{a}^\dagger - \frac{1}{2} \{\hat{\rho}, \hat{a}^\dagger \hat{a}\} \right] + \eta N_\beta(\omega) \left[ \hat{a}^\dagger \hat{\rho} \hat{a} - \frac{1}{2} \{\hat{\rho}, \hat{a} \hat{a}^\dagger\} \right],$$

i.e., the well-known quantum optical master equation for the description of a damped harmonic oscillator (for a recent review, see Ref. 30), where the only free parameter is the decay rate  $\eta$  and a further freedom appears in the commutator term, where a function of  $\hat{N}$  corresponding to a frequency shift may be considered. The quantum optical master equation is therefore essentially fixed by formal requirements, well in accordance with its stability with respect to microphysical derivations, which are in fact predictive in so far as they give explicit expressions for  $\eta$  and the energy shift. As a last step, using as natural length of the problem  $l = \sqrt{\hbar/M\omega}$ , (10) may be written in terms of  $\hat{x}$  and  $\hat{p}$  as

$$\begin{aligned} \mathcal{M}_{\hat{x}\hat{p}}^{\text{QO}}[\hat{\rho}] = & -\frac{i}{\hbar} \left[ \frac{\hat{p}^2}{2M} + \frac{1}{2} M \omega^2 \hat{x}^2, \hat{\rho} \right] - \frac{i \gamma}{\hbar 2} (\{ \hat{x}, \{ \hat{p}, \hat{\rho} \} \} - \{ \hat{p}, \{ \hat{x}, \hat{\rho} \} \}) \\ & - \frac{1 \gamma}{\hbar 2} \coth\left(\frac{1}{2} \beta \hbar \omega\right) \left( M \omega \{ \hat{x}, [\hat{x}, \hat{\rho}] \} + \frac{1}{M \omega} [\hat{p}, [\hat{p}, \hat{\rho}]] \right), \end{aligned}$$

where invariance under (9) can be easily checked.

That U(1) symmetry or shift-covariance may lead, under suitable restrictions, to the quantum optical master equation can also be seen considering the recently obtained most general structure of a proper generator of a shift-covariant completely positive quantum dynamical semigroup given in Ref. 29, where also the unboundedness of the operators appearing in the formal Lindblad structure has been taken in due account, using the notion of form-generator. The formal operator expression associated to the form-generator is

$$\begin{aligned} \mathcal{L}[\hat{\rho}] = & -\frac{i}{\hbar} [H(\hat{N}), \hat{\rho}] + \left[ A_0(\hat{N}) \hat{\rho} A_0^\dagger(\hat{N}) - \frac{1}{2} \{ \hat{\rho}, A_0^\dagger(\hat{N}) A_0(\hat{N}) \} \right] \\ & + \sum_{m=1}^{\infty} \left[ \hat{W}^{\dagger m} A_{-m}(\hat{N}) \hat{\rho} A_{-m}^\dagger(\hat{N}) \hat{W}^m - \frac{1}{2} \{ \hat{\rho}, A_{-m}^\dagger(\hat{N}) \hat{W}^m \hat{W}^{\dagger m} A_{-m}(\hat{N}) \} \right] \\ & + \sum_{m=1}^{\infty} \left[ \hat{W}^m A_m(\hat{N}) \hat{\rho} A_m^\dagger(\hat{N}) \hat{W}^{\dagger m} - \frac{1}{2} \{ \hat{\rho}, A_m^\dagger(\hat{N}) A_m(\hat{N}) \} \right], \end{aligned} \tag{11}$$

where  $H(\cdot) = H^*(\cdot)$ ,  $A_i(\cdot)$  are functions of the number operator, and  $\hat{W} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$  is the so-called shift-operator.<sup>31</sup> Recalling the polar decompositions  $\hat{a}^\dagger = \hat{W} \sqrt{\hat{N}+1}$  and  $\hat{a} = \hat{W}^\dagger \sqrt{\hat{N}}$ , for the following simple choice of functions,

$$\begin{aligned} H(n) = H_0(n) &= \hbar \omega \left( n + \frac{1}{2} \right), \\ A_m(n) &= 0, \quad m=0, \quad |m| > 1, \\ A_1(n) &= \sqrt{\gamma(\beta)} \sqrt{n+1}, \\ A_{-1}(n) &= e^{\beta/2} \sqrt{\gamma(\beta)} \sqrt{n}, \end{aligned}$$

where

$$\gamma(\beta) = \gamma \left[ \coth \left( \frac{1}{2} \beta \hbar \omega \right) - 1 \right],$$

one recovers from (11) the quantum optical master equation.

It is interesting to observe that complete positivity, shift-covariance and the requirement of a canonical stationary solution also allow as a proper generator of a quantum dynamical semigroup the following expression:

$$\begin{aligned} \mathcal{L}[\hat{\rho}] = & -\frac{i}{\hbar} [H_0(\hat{N}), \hat{\rho}] - \gamma_0 [\hat{N}, [\hat{N}, \hat{\rho}]] \\ & + \sum_{m=1}^{\infty} \gamma_m \left\{ \left[ \coth \left( \frac{1}{2} \beta \hbar \omega \right) + 1 \right]^m \left[ \hat{a}^m \hat{\rho} \hat{a}^{\dagger m} - \frac{1}{2} \{ \hat{\rho}, \hat{a}^{\dagger m} \hat{a}^m \} \right] \right. \\ & \left. + \left[ \coth \left( \frac{1}{2} \beta \hbar \omega \right) - 1 \right]^m \left[ \hat{a}^{\dagger m} \hat{\rho} \hat{a}^m - \frac{1}{2} \{ \hat{\rho}, \hat{a}^m \hat{a}^{\dagger m} \} \right] \right\}, \end{aligned} \tag{12}$$

corresponding to the functions

$$\begin{aligned} A_0(n) &= \sqrt{\gamma_0 n}, \\ A_m(n) &= \sqrt{\gamma_m(\beta)} \sqrt{\frac{(n+m)!}{n!}}, \\ A_{-m}(n) &= e^{m\beta/2} \sqrt{\gamma_m(\beta)} \sqrt{\frac{n!}{(n-m)!}}, \end{aligned}$$

where

$$\gamma_m(\beta) = \gamma_m \left[ \coth \left( \frac{1}{2} \beta \hbar \omega \right) - 1 \right]^m.$$

Equation (12) provides a generalization of the quantum optical master equation in which both phase-diffusion, related to the coefficient  $\gamma_0$ , and  $m$ -photon processes with decay rates  $\eta_m = 2^m \gamma_m$ , which should quickly approach zero in order to allow for nonexplosion of the associated Markov process, can be considered. Thus far we have dealt with the case of shift-covariance, corresponding to U(1) symmetry of the system with many degrees of freedom acting as reservoir and determining the non-Hamiltonian dynamics of the microsystem.

**B. Translation-covariance**

We now consider the case in which the relevant symmetry is invariance under translations, corresponding to a homogeneous reservoir. Given the unitary representation of the translation group

$$\hat{U}(b) = e^{- (i/\hbar) b \hat{p}},$$

where  $\hat{p}$  is the momentum operator of the microsystem and  $b \in \mathbb{R}$ , translation-covariance according to Ref. 32 amounts to the requirement

$$\mathcal{L}_{\hat{x}\hat{p}}[e^{- (i/\hbar) b \hat{p}} \cdot e^{(i/\hbar) b \hat{p}}] = e^{- (i/\hbar) b \hat{p}} \mathcal{L}_{\hat{x}\hat{p}}[\cdot] e^{(i/\hbar) b \hat{p}}. \tag{13}$$

Invariance under the group  $\mathbb{R}$  of translations thus implies for the structure of the quantum Fokker–Planck equation that the Hamiltonian has to be a function of the generator of the transformation  $\hat{p}$  and the following simple requirement in the coefficients appearing in (2):

$$\mu = \gamma.$$

Considering, for example, a free particle interacting with a homogeneous reservoir, one has  $\mathbb{R}$  symmetry corresponding to invariance under translations, which is reflected by the fact that according to (13) the master equation is invariant under the transformation

$$\hat{x} \rightarrow \hat{x} + b, \quad \hat{p} \rightarrow \hat{p},$$

or, equivalently, in terms of  $\hat{a}$  and  $\hat{a}^\dagger$ ,

$$\hat{a} \rightarrow \hat{a} + \frac{1}{\sqrt{2}l} b, \quad \hat{a}^\dagger \rightarrow \hat{a}^\dagger + \frac{1}{\sqrt{2}l} b.$$

Further, making the obvious choice  $H_0(\hat{p}) = \hat{p}^2/2M$ , corresponding to a free particle of mass  $M$  and asking that an operator with the canonical structure  $\hat{\rho}_0 = e^{-\beta H_0(\hat{p})}$  be a stationary solution, i.e.,  $\mathcal{L}_{\hat{x}\hat{p}}[\hat{\rho}_0] = 0$ , one has the condition

$$D_{pp} = \gamma \frac{2M}{\beta},$$

with  $\beta$  the inverse temperature of the homogeneous reservoir. The requirement, of complete positivity, translation-covariance and existence of the expected stationary solution thus constrain the quantum Fokker–Planck equation to be of the form

$$\mathcal{L}_{\hat{x}\hat{p}}^{\text{QBM}}[\hat{\rho}] = - \frac{i}{\hbar} [H_0(\hat{p}), \hat{\rho}] - \frac{i}{\hbar} \gamma [\hat{x}, \{\hat{p}, \hat{\rho}\}] - \gamma \frac{2M}{\beta \hbar^2} [\hat{x}, [\hat{x}, \hat{\rho}]] - \frac{D_{xx}}{\hbar^2} [\hat{p}, [\hat{p}, \hat{\rho}]] + 2 \frac{D_{px}}{\hbar^2} [\hat{x}, [\hat{p}, \hat{\rho}]], \tag{14}$$

i.e., the typical structure one finds in the extensive physical literature aiming at the description of quantum Brownian motion (see Refs. 1 and 3 for a review). With respect to the case of the

quantum optical master equation (10), the remaining freedom in the structure is much bigger: apart from the friction coefficient  $\gamma$  and a real function of  $\hat{p}$  correcting the Hamiltonian, the coefficients  $D_{xx}$  and  $D_{px}$  are undetermined except for the relation

$$\gamma \frac{2M}{\beta} D_{xx} - D_{px}^2 \geq \frac{\gamma^2 \hbar^2}{4}$$

stemming from (3). This is reflected by the fact that a much wider literature has been devoted to the subject, looking for microphysical derivations of the quantum Brownian motion master equation, in order to obtain expressions for the undetermined parameters.

We now compare the result (14) with the general structure of a proper generator of a translation-covariant quantum dynamical semigroup, given in Ref. 32, where also the case of an unbounded generator has been considered, introducing the notion of form-generator and specifying a suitable domain for the mapping. Restricting to the continuous component of the generator, corresponding to a quantum Fokker–Planck equation describing friction and diffusion, one has for the formal operator expression associated to the form-generator the following result:

$$\mathcal{L}[\hat{\rho}] = -\frac{i}{\hbar} [\beta \hat{x} + H(\hat{p}), \hat{\rho}] + \alpha \hat{V} \hat{\rho} \hat{V}^\dagger - \hat{K} \hat{\rho} - \hat{\rho} \hat{K}^\dagger, \quad \beta \in \mathbb{R}, \quad \alpha \geq 0, \tag{15}$$

$$\hat{V} = \hat{x} + L(\hat{p}), \quad \hat{K} = \frac{\alpha}{2} [\hat{x}^2 + 2\hat{x}L(\hat{p}) + L^\dagger(\hat{p})L(\hat{p})],$$

$H(\cdot) = H^*(\cdot)$ ,  $L(\cdot)$  being functions of the momentum operator  $\hat{p}$ , and  $\beta \neq 0$  implying, e.g., a constant gravitational or electric field. According to (15) one has a single operator of the form  $\hat{V} = \hat{x} + L(\hat{p})$  (or one for each Cartesian coordinate considering higher dimensions) instead of two as considered in (1). Expressing (15) in terms of nested commutators and anticommutators as in (2), restricting to the case in which  $L(\cdot)$  is a linear function, according to the fact that we are considering friction effects at most linear in the velocity, the inequalities in (3), corresponding to the fact that the determinant of the matrix given in (4) be zero or positive, now become more restrictive:

$$D_{xx} \geq 0, \quad D_{pp} \geq 0, \quad D_{xx}D_{pp} - D_{px}^2 = \frac{\gamma^2 \hbar^2}{4},$$

corresponding to  $\det \mathbf{D} = 0$ . Coming back to (14), this implies the further restriction

$$D_{xx} = \gamma \frac{\beta \hbar^2}{8M} + \frac{\beta}{2\gamma M} D_{px}^2,$$

so that apart from the overall multiplying coefficient  $\gamma$  only another coefficient  $D_{px}$  is left free, and one has

$$\begin{aligned} \mathcal{L}_{\hat{x}\hat{p}}^{\text{QBM}}[\hat{\rho}] = & -\frac{i}{\hbar} [H_0(\hat{p}), \hat{\rho}] - \frac{i}{\hbar} \gamma [\hat{x}, \{\hat{p}, \hat{\rho}\}] - \gamma \frac{2M}{\beta \hbar^2} [\hat{x}, [\hat{x}, \hat{\rho}]] - \gamma \frac{\beta}{8M} [\hat{p}, [\hat{p}, \hat{\rho}]] \\ & - \frac{\beta}{2\gamma M} \frac{D_{px}^2}{\hbar^2} [\hat{p}, [\hat{p}, \hat{\rho}]] + 2 \frac{D_{px}}{\hbar^2} [\hat{p}, [\hat{x}, \hat{\rho}]]. \end{aligned} \tag{16}$$

Therefore, a predictive microphysical model of quantum Brownian motion essentially has to indicate an explicit expression for the coefficients  $\gamma$  and  $D_{px}$ .

It is interesting to express (16) in terms of the creation and annihilation operators given in (5). Setting  $\beta \hbar^2 / 4M = \lambda_{\text{th}}^2$ , the square of the thermal wavelength of the microsystem undergoing Brownian motion, one has

$$\begin{aligned}
 \mathcal{L}_{\hat{a}\hat{a}^\dagger}^{\text{QBM}}[\hat{\rho}] = & -\frac{i}{\hbar}[H_0(\hat{a},\hat{a}^\dagger),\hat{\rho}] - \frac{\gamma}{2}[\hat{a}^2 - \hat{a}^{\dagger 2},\hat{\rho}] \\
 & + \left[ \frac{\gamma}{2} \left( \frac{\lambda_{\text{th}}^2}{l^2} + \frac{l^2}{\lambda_{\text{th}}^2} + 2 \right) + \frac{1}{\gamma \hbar^2} \frac{2\lambda_{\text{th}}^2}{l^2} D_{px}^2 \right] \left[ \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\{\hat{\rho},\hat{a}^\dagger\hat{a}\} \right] \\
 & + \left[ \frac{\gamma}{2} \left( \frac{\lambda_{\text{th}}^2}{l^2} + \frac{l^2}{\lambda_{\text{th}}^2} - 2 \right) + \frac{1}{\gamma \hbar^2} \frac{2\lambda_{\text{th}}^2}{l^2} D_{px}^2 \right] \left[ \hat{a}^\dagger\hat{\rho}\hat{a} - \frac{1}{2}\{\hat{\rho},\hat{a}\hat{a}^\dagger\} \right] \\
 & - \left[ \frac{\gamma}{2} \left( \frac{\lambda_{\text{th}}^2}{l^2} - \frac{l^2}{\lambda_{\text{th}}^2} \right) + 2 \frac{D_{px}}{\hbar} \left( \frac{1}{\gamma l^2} \frac{D_{px}}{\hbar} - i \right) \right] \left[ \hat{a}\hat{\rho}\hat{a} - \frac{1}{2}\{\hat{\rho},\hat{a}^2\} \right] + \text{h.c.} \quad (17)
 \end{aligned}$$

Equation (17) strongly simplifies if one takes for the length  $l$ , used in order to introduce the operators  $\hat{a}$  and  $\hat{a}^\dagger$  in terms of the operator position and momentum of the particle, the value  $l = \lambda_{\text{th}} = \sqrt{\beta \hbar^2 / 4M}$ , naturally suggested by the underlying physics. With  $l$  the thermal de Broglie wavelength (17) reduces to

$$\begin{aligned}
 \mathcal{L}_{\hat{a}\hat{a}^\dagger}^{\text{QBM}}[\hat{\rho}] = & -\frac{i}{\hbar}[H_0(\hat{a},\hat{a}^\dagger),\hat{\rho}] - \frac{\gamma}{2}[\hat{a}^2 - \hat{a}^{\dagger 2},\hat{\rho}] + 2\gamma \left[ \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\{\hat{\rho},\hat{a}^\dagger\hat{a}\} \right] \\
 & + \frac{2D_{px}^2}{\gamma \hbar^2} \left[ \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\{\hat{\rho},\hat{a}^\dagger\hat{a}\} + \hat{a}^\dagger\hat{\rho}\hat{a} - \frac{1}{2}\{\hat{\rho},\hat{a}\hat{a}^\dagger\} \right] \\
 & - 2 \frac{D_{px}}{\hbar} \left( \frac{1}{\gamma} \frac{D_{px}}{\hbar} - i \right) \left[ \hat{a}\hat{\rho}\hat{a} - \frac{1}{2}\{\hat{\rho},\hat{a}^2\} \right] + \text{h.c.}, \quad (18)
 \end{aligned}$$

where the last three contributions can only vanish if the real coefficient  $D_{px}$  is equal to zero, corresponding to  $L(\cdot) = -L^*(\cdot)$  in (15). Equation (16) or equivalently (18) expresses the general structure of a quantum Fokker–Planck equation which is invariant under translation, warrants the existence of the expected canonical expression as a stationary solution, thus recovering equipartition, and is furthermore a proper generator of a completely positive quantum dynamical semi-group.

### C. Covariance and uniqueness of the stationary solution

In recent work<sup>2,12</sup> aiming at comparing and clarifying different approaches to quantum dissipation, which relies on<sup>14</sup> but neglects the more recent and thorough results of Refs. 32 and 29, the statement can be found that *no Markovian theory can satisfy all three criteria of positivity, translational invariance, and asymptotic approach to the canonical equilibrium state  $e^{-\beta H_0}$ , except in special cases.* This statement is always correct in view of the last observation, and these simple exceptional cases, which are usually not spelled out, can be read in (11): the microsystem has to be a free particle apart from an effective correction to the Hamiltonian, given by a real function of  $\hat{p}$  describing, for example, an effective mass, and a potential term depending linearly on position (such as, e.g., a constant gravitational or electric field). These cases are often neglected, having in mind that translational invariance, mainly seen as an abstract property rather than the expression of homogeneity of the reservoir, should be always asked for. In this spirit translational invariance, i.e., R symmetry, is asked for also for the damped harmonic oscillator.

The three physical requirements one can reasonably ask together for Markovian systems in the weak-coupling limit are complete positivity, existence of the stationary solution predicted by equipartition, and invariance under the relevant symmetry, not necessarily translational invariance. Having translational invariance apart from the potential term is not physically significant since the potential term actually breaks this invariance, and would furthermore lead to high nonuniqueness of the stationary state, as argued below. The dynamics of the microsystem is driven by both the potential term (which could also arise as a mean field effect) and the contributions describing

decoherence and dissipation, so that a physically relevant symmetry should pertain to the system as a whole. Let us in fact consider a mapping  $\mathcal{F}$  covariant with respect to a given symmetry group  $G$  according to (7), which admits a stationary solution  $\hat{\rho}_0$ , i.e.,  $\mathcal{F}[\hat{\rho}_0]=0$ . If the operator  $\hat{\rho}_0$  is not invariant under the unitary representation  $\mathcal{U}_g$  of  $G$ , so that

$$\hat{\rho}_g = \mathcal{U}_g[\hat{\rho}_0]$$

is linearly independent from  $\hat{\rho}_0$  at least for some  $g$  in  $G$ , due to the  $G$ -covariance of  $\mathcal{F}$ ,  $\hat{\rho}_g$  still is a stationary solution

$$\mathcal{F}[\hat{\rho}_g] = \mathcal{F}[\mathcal{U}_g[\hat{\rho}_0]] = \mathcal{U}_g[\mathcal{F}[\hat{\rho}_0]] = 0,$$

so that one cannot have the expected uniqueness of the solution. In the case of the one-dimensional Lie groups considered in Sec. II, for example, a stationary solution can be unique only if it commutes with the generator of the group. Note that this simple argument is independent on whether the mapping  $\mathcal{F}$  ensures complete positivity of the time evolution or not, so that the clash between the requirement of translation invariance and the existence of the correct stationary state corresponding to equipartition is not due to the requirement of complete positivity of the time evolution mapping.

### III. QUANTUM FOKKER–PLANCK EQUATION FOR THE MOTION OF A BROWNIAN PARTICLE

As we have shown in Sec. II A, in the case of an underlying U(1) symmetry formal requirements are enough to essentially fix structure and coefficients of the quantum Fokker–Planck equation, apart from an energy shift and an overall coefficient. The same is not true in the case of R symmetry describing translational invariance. In this paragraph we therefore quickly recall a recently proposed quantum Fokker–Planck equation for the description of the motion of a heavy Brownian particle interacting through collisions with a homogeneous fluid made up of much lighter particles (see Refs. 20, 21, and 23 for details). This result has been obtained within a kinetic approach, where the dynamics is driven by single events described as collisions in which one generally has momentum and energy transfer, alternative to the Zwanzig Caldeira Leggett approach,<sup>33</sup> recently criticized in Ref. 34, where one describes the reservoir as a collection of harmonic oscillators coupled to the microsystem through its position, usually performing calculations in terms of path-integral techniques.

In the aforementioned approach one obtains in the first instance a kinetic equation for the statistical operator analogous to the classical linear Boltzmann equation given by the simple expression

$$\begin{aligned} \mathcal{L}[\hat{\rho}] = & -\frac{i}{\hbar} [H_0(\hat{\mathbf{p}}), \hat{\rho}] + \frac{2\pi}{\hbar} (2\pi\hbar)^3 n \int_{\mathbf{R}^3} d^3\mathbf{q} |\tilde{t}(q)|^2 \left[ e^{(i/\hbar)\mathbf{q}\cdot\hat{\mathbf{x}}\sqrt{S(\mathbf{q},\hat{\mathbf{p}})}\hat{\rho}\sqrt{S(\mathbf{q},\hat{\mathbf{p}})} e^{-(i/\hbar)\mathbf{q}\cdot\hat{\mathbf{x}}} \right. \\ & \left. - \frac{1}{2} \{S(\mathbf{q},\hat{\mathbf{p}}), \hat{\rho}\} \right], \end{aligned} \tag{19}$$

where  $\tilde{t}(q)$  is the Fourier transform of the T-matrix describing the microphysical collisions and the function  $S(\mathbf{q},\mathbf{p})$  appearing operator-valued in (19) is a positive two-point correlation function known in the physical community as dynamic structure factor,<sup>35</sup> usually expressed as a function of momentum and energy transfer,  $\mathbf{q}$  and  $E$ , according to

$$S(\mathbf{q},\mathbf{p}) \equiv S(\mathbf{q},E), \quad E(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})^2}{2M} - \frac{\mathbf{p}^2}{2M},$$

with  $M$  the mass of the Brownian particle. The dynamic structure factor  $S(\mathbf{q},E)$  is the Fourier transform of the two-point time dependent density autocorrelation function of the fluid

$$S(\mathbf{q}, E) = \frac{1}{2\pi\hbar} \int_{\mathbf{R}} dt \int_{\mathbf{R}^3} d^3\mathbf{x} e^{(i/\hbar)[E(\mathbf{q}, \mathbf{p})t - \mathbf{q} \cdot \mathbf{x}]} \frac{1}{N} \int_{\mathbf{R}^3} d^3\mathbf{y} \langle N(\mathbf{y})N(\mathbf{y} + \mathbf{x}, t) \rangle, \quad (20)$$

and is always positive since it is proportional to the energy dependent scattering cross-section of a microscopic probe off a macroscopic sample,<sup>36</sup> giving the spectrum of its spontaneous fluctuations. Equation (19) also has the three properties of complete positivity, translational invariance and canonical stationary solution, and it actually gives a physical example of the general structure of generator of a translation-covariant quantum dynamical semigroup,<sup>32</sup> going beyond the diffusive case considered in (11). In order to recover from the general integral kinetic Eq. (19) a quantum Fokker–Planck equation for the description of the Brownian motion of a heavy particle, one has to consider the limit of small momentum transfer and small energy transfer (corresponding to the Brownian limit in which the mass of the test particle is much heavier than the particles of the fluid). Considering a gas of free particles obeying Maxwell–Boltzmann statistics, due to

$$S_{\text{MB}}(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^3} \frac{2\pi m^2}{n\beta q} z \exp\left[-\frac{\beta}{8m} \frac{(2mE(\mathbf{q}, \mathbf{p}) + q^2)^2}{q^2}\right] \quad (21)$$

one has<sup>23</sup>

$$\mathcal{L}[\hat{\rho}] = -\frac{i}{\hbar}[H_0(\hat{\rho}), \hat{\rho}] - \frac{i}{\hbar} \gamma \sum_{i=1}^3 [\hat{x}_i, \{\hat{\rho}_i, \hat{\rho}\}] - \frac{D_{pp}}{\hbar^2} \sum_{i=1}^3 [\hat{x}_i, [\hat{x}_i, \hat{\rho}]] - \frac{D_{xx}}{\hbar^2} \sum_{i=1}^3 [\hat{p}_i, [\hat{p}_i, \hat{\rho}]]. \quad (22)$$

In this kinetic case the free parameters in (16) are now determined as

$$D_{px} = 0, \quad \gamma = \frac{1}{3} z \frac{\pi^2 m^2}{\beta \hbar} \int_{\mathbf{R}^3} d^3\mathbf{q} |\tilde{t}(q)|^2 q e^{-(\beta/8m) q^2}$$

with  $z = e^{\beta\mu}$  the fugacity of the gas,<sup>37</sup> while  $D_{xx}$  and  $D_{pp}$  are expressed in terms of  $\gamma$  as can be read in (16):

$$D_{xx} = \frac{\beta \hbar^2}{8M} \gamma, \quad D_{pp} = \frac{2M}{\beta} \gamma.$$

Also the expression of (22) in terms of the operators  $\hat{\mathbf{a}}_i$  and  $\hat{\mathbf{a}}_i^\dagger$  according to (18) takes in this case a particularly simple form

$$\mathcal{L}[\hat{\rho}] = -\frac{i}{\hbar}[H_0(\hat{\mathbf{a}}_i, \hat{\mathbf{a}}_i^\dagger), \hat{\rho}] - \frac{\gamma}{2} \sum_{i=1}^3 [\hat{\mathbf{a}}_i^2 - \hat{\mathbf{a}}_i^{\dagger 2}, \hat{\rho}] + 2\gamma \sum_{i=1}^3 \left[ \hat{\mathbf{a}}_i \hat{\rho} \hat{\mathbf{a}}_i^\dagger - \frac{1}{2} \{\hat{\rho}, \hat{\mathbf{a}}_i^\dagger \hat{\mathbf{a}}_i\} \right], \quad (23)$$

so that one has a single  $\hat{\mathbf{a}}_i = \sqrt{2M/\beta\hbar^2}(\hat{x}_i + i(\beta\hbar^2/4M)\hat{p}_i)$  operator for each Cartesian coordinate.

#### IV. CONCLUSIONS AND OUTLOOK

The main scope of this article was to show how relevant symmetries can be in the determination of structures of quantum Fokker–Planck equation. While in the literature only translational invariance is considered and asked for also in the case of the damped harmonic oscillator, leading to no go statements regarding the possibility of having quantum Fokker–Planck equations with all the physically relevant features (complete positivity, covariance and equipartition), we here considered both one-dimensional Lie groups U(1) and R, showing that they are connected to two distinct classes of quantum Fokker–Planck equations: the quantum optical master equation associated to shift-covariance and the quantum Brownian motion master equation associated to translation-covariance. That these two classes of models actually correspond to different physics can also be seen in connection with recent studies on their properties with respect to decoherence.<sup>8</sup>

In particular, independently of complete positivity of the time evolution, it has been shown that covariance properties put severe restrictions on the structure of the stationary solution, provided uniqueness is asked for.

In the case of shift-covariance a generalization of the quantum optical master equation has been proposed, in which also  $m$ -photon processes can be considered. Moreover, recent mathematical results by Holevo have been considered, concerning the structure of generators of completely positive quantum dynamical semigroups and leading to further restrictions in the quantum Brownian motion case. For the Brownian motion of a test particle in a fluid, where formal requirements are not enough to essentially fix the structure of the quantum Fokker–Planck equation describing the phenomenon, a recent microphysical approach has been briefly recalled, based on scattering theory, where the quantum Fokker–Planck equation is obtained as the small momentum and energy transfer limit of a quantum generalization of the classical linear Boltzmann equation. Covariance properties with respect to some physically relevant group, typically reflecting a symmetry under certain transformations of the given reservoir, can therefore be a most useful requirement in the determination of structures of quantum Fokker–Planck equations or more generally of linear kinetic equations.

An interesting extension of this work would entail the study of generators of the dynamics of systems in which one has an important correlation between internal and translational degrees of freedom, both coupled through different interactions to some reservoir, a problem recently considered in Ref. 38, where the translational degrees of freedom are treated in a classical way, assuming decoherence is strong enough. Such models could be of interest for the implementation of quantum computing, where indeed some experimental scheme actually relies on a coupling between internal and center of mass degrees of freedom.<sup>39</sup>

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