INTRODUCTION

§ 1. The uncertainty principle in the relativistic case

THE quantum theory described in Volume 3 (*Quantum Mechanics*) is essentially non-relativistic throughout, and is not applicable to phenomena involving motion at velocities comparable with that of light. At first sight, one might expect that the change to a relativistic theory is possible by a fairly direct generalization of the formalism of non-relativistic quantum mechanics. But further consideration shows that a logically complete relativistic theory cannot be constructed without invoking new physical principles.

Let us recall some of the physical concepts forming the basis of non-relativistic quantum mechanics (*QM*, §1). We saw that one fundamental concept is that of *measurement*, by which is meant the process of interaction between a quantum system and a classical object or *apparatus*, causing the quantum system to acquire definite values of some particular dynamical variables (coordinates, velocities, etc.). We saw also that quantum mechanics greatly restricts the possibility that an electron† simultaneously possesses values of different dynamical variables. For example, the uncertainties Δq and Δp in simultaneously existing values of the coordinate and the momentum are related by the expression‡ $\Delta q \Delta p \sim \hbar$; the greater the accuracy with which one of these quantities is measured, the less the accuracy with which the other can be measured at the same time.

It is important to note, however, that any of the dynamical variables of the electron can individually be measured with arbitrarily high accuracy, and in an arbitrarily short period of time. This fact is of fundamental importance throughout non-relativistic quantum mechanics. It is the only justification for using the concept of the wave function, which is a basic part of the formalism. The physical significance of the wave function $\psi(q)$ is that the square of its modulus gives the probability of finding a particular value of the electron coordinate as the result of a measurement made at a given instant. The concept of such a probability clearly requires that the coordinate can in principle be measured with any specified accuracy and rapidity, since otherwise this concept would be purposeless and devoid of physical significance.

The existence of a limiting velocity (the velocity of light, denoted by c) leads to new fundamental limitations on the possible measurements of various physical quantities (L. D. Landau and R. E. Peierls, 1930).

† As in *QM*, §1, we shall, for brevity, speak of an "electron", meaning any quantum system.
‡ In this section, ordinary units are used.

In *QM*, §44, the following relationship has been derived:

$$(v' - v)\Delta p \Delta t \sim \hbar, \tag{1.1}$$

relating the uncertainty Δp in the measurement of the electron momentum and the duration Δt of the measurement process itself; v and v' are the velocities of the electron before and after the measurement. From this relationship it follows that a momentum measurement of high accuracy made during a short time (i.e. with Δp and Δt both small) can occur only if there is a large change in the velocity as a result of the measurement process itself. In the non-relativistic theory, this showed that the measurement of momentum cannot be repeated at short intervals of time, but it did not at all diminish the possibility, in principle, of making a single measurement of the momentum with arbitrarily high accuracy, since the difference $v' - v$ could take any value, no matter how large.

The existence of a limiting velocity, however, radically alters the situation. The difference $v' - v$, like the velocities themselves, cannot now exceed c (or rather $2c$). Replacing $v' - v$ in (1.1) by c, we obtain

$$\Delta p \Delta t \sim \hbar / c, \tag{1.2}$$

which determines the highest accuracy theoretically attainable when the momentum is measured by a process occupying a given time Δt. In the relativistic theory, therefore, it is in principle impossible to make an arbitrarily accurate and rapid measurement of the momentum. An exact measurement ($\Delta p \to 0$) is possible only in the limit as the duration of the measurement tends to infinity.

There is reason to suppose that the concept of measurability of the electron coordinate itself must also undergo modification. In the mathematical formalism of the theory, this situation is shown by the fact that an accurate measurement of the coordinate is incompatible with the assertion that the energy of a free particle is positive. It will be seen later that the complete set of eigenfunctions of the relativistic wave equation of a free particle includes, as well as solutions having the "correct" time dependence, also solutions having a "negative frequency". These functions will in general appear in the expansion of the wave packet corresponding to an electron localized in a small region of space.

It will be shown that the wave functions having a "negative frequency" correspond to the existence of antiparticles (positrons). The appearance of these functions in the expansion of the wave packet expresses the (in general) inevitable production of electron–positron pairs in the process of measuring the coordinates of an electron. This formation of new particles in a way which cannot be detected by the process itself renders meaningless the measurement of the electron coordinates.

In the rest frame of the electron, the least possible error in the measurement of its coordinates is

$$\Delta q \sim \hbar / mc. \tag{1.3}$$

This value (which purely dimensional arguments show to be the only possible one) corresponds to a momentum uncertainty $\Delta p \sim mc$, which in turn corresponds to the threshold energy for pair production.

In a frame of reference in which the electron is moving with energy ε, (1.3) becomes

$$\Delta q \sim c\hbar/\varepsilon. \tag{1.4}$$

In particular, in the limiting ultra-relativistic case the energy is related to the momentum by $\varepsilon \approx cp$, and

$$\Delta q \sim \hbar/p, \tag{1.5}$$

i.e. the error Δq is the same as the de Broglie wavelength of the particle.†

For photons, the ultra-relativistic case always applies, and the expression (1.5) is therefore valid. This means that the coordinates of a photon are meaningful only in cases where the characteristic dimensions of the problem are large in comparison with the wavelength. This is just the "classical" limit, corresponding to geometrical optics, in which radiation can be said to be propagated along definite paths or rays. In the quantum case, however, where the wavelength cannot be regarded as small, the concept of coordinates of the photon has no meaning. We shall see later (§4) that, in the mathematical formalism of the theory, the fact that the photon coordinates cannot be measured is evident because the photon wave function cannot be used to construct a quantity which might serve as a probability density satisfying the necessary conditions of relativistic invariance.

The foregoing discussion suggests that the theory will not consider the time dependence of particle interaction processes. It will show that in these processes there are no characteristics precisely definable (even within the usual limitations of quantum mechanics); the description of such a process as occurring in the course of time is therefore just as unreal as the classical paths are in non-relativistic quantum mechanics. The only observable quantities are the properties (momenta, polarizations) of free particles: the initial particles which come into interaction, and the final particles which result from the process (L. D. Landau and R. E. Peierls, 1930).

A typical problem as formulated in relativistic quantum theory is to determine the probability amplitudes of transitions between specified initial and final states ($t \to \mp\infty$) of a system of particles. The set of such amplitudes between all possible states constitutes the *scattering matrix* or *S-matrix*. This matrix will embody all the information about particle interaction processes that has an observable physical meaning (W. Heisenberg, 1938).

There is as yet no logically consistent and complete relativistic quantum theory. We shall see that the existing theory introduces new physical features into the nature of the description of particle states, which acquires some of the features of

† The measurements in question are those for which any experimental result yields a conclusion about the state of the electron; that is, we are not considering coordinate measurements by means of collisions, when the result does not occur with probability unity during the time of observation. Although the deflection of a measuring-particle in such cases may indicate the position of an electron, the absence of a deflection tells us nothing.

field theory (see §10). The theory is, however, largely constructed on the pattern of ordinary quantum mechanics. This structure of the theory has yielded good results in quantum electrodynamics. The lack of complete logical consistency in this theory is shown by the occurrence of divergent expressions when the mathematical formalism is directly applied, although there are quite well-defined ways of eliminating these divergences. Nevertheless, such methods remain, to a considerable extent, semiempirical rules, and our confidence in the correctness of the results is ultimately based only on their excellent agreement with experiment, not on the internal consistency or logical ordering of the fundamental principles of the theory.