

Chapter 10

The Electromagnetic Interaction

In this chapter we will examine the electroweak interaction of the standard model, and, in particular, the electromagnetic part of it. We relegate the weak part to the next chapter. The electromagnetic interaction is important in subatomic physics for two reasons. First, it enters whenever a charged particle is used as a probe. Second, it is the only interaction whose form can be studied in classical physics, and it provides a model after which other interactions can be patterned.

Without at least some approximate computations, interactions cannot be understood. In the simplest form, such computations are based on quantum mechanical perturbation theory and, in particular, on the expression for the transition rate from an initial state α to a final state β :

$$w_{\beta\alpha} = \frac{2\pi}{\hbar} |\langle\beta|H_{\text{int}}|\alpha\rangle|^2 \rho(E). \quad (10.1)$$

Fermi called this expression *the golden rule*, because of its usefulness and importance. In Section 10.1 we shall derive this relation; in Section 10.2, we shall discuss the density-of-states factor $\rho(E)$. Readers who are familiar with these topics can omit these two sections.

10.1 The Golden Rule

Consider a system that is described by a time-independent Hamiltonian H_0 ; its Schrödinger equation is

$$i\hbar \frac{\partial \varphi}{\partial t} = H_0 \varphi. \quad (10.2)$$

The stationary states of this system are found by inserting the ansatz,

$$\varphi = u_n(\mathbf{x}) \exp\left(\frac{-iE_n t}{\hbar}\right) \quad (10.3)$$

into Eq. (10.2). The result is the time-independent Schrödinger equation

$$H_0 u_n = E_n u_n. \quad (10.4)$$

For the further discussion it is assumed that this equation has been solved, that the eigenvalues E_n and the eigenfunctions u_n are known, and that the eigenfunctions form a complete orthonormal set, with

$$\int d^3x u_N^*(\mathbf{x}) u_n(\mathbf{x}) = \delta_{Nn}. \quad (10.5)$$

If the system is produced in one of the eigenstates u_n , it will remain in that state forever and no transitions to other states will occur.

We next consider a system that is similar to the one just discussed, but its Hamiltonian, H , differs from H_0 by a small term, the interaction Hamiltonian, H_{int} ,

$$H = H_0 + H_{\text{int}}.$$

The state of this system can, in zeroth approximation, still be characterized by the energies E_n and the eigenfunctions u_n . It is still possible to form the system in a state described by one of the eigenfunctions u_n , and we shall call a particular initial state $|\alpha\rangle$.

However, such a state will in general no longer be stationary; the perturbing Hamiltonian H_{int} will cause transitions to other states, for instance, $|\beta\rangle$. In the following we shall derive an expression for the transition rate $|\alpha\rangle \rightarrow |\beta\rangle$. Two examples of such transitions are shown in Fig. 10.1. In Fig. 10.1(a), the interaction is responsible for the decay of the state via the emission of a photon. In Fig. 10.1(b), an incident particle in state $|\alpha\rangle$ is scattered into the state $|\beta\rangle$.

To compute the rate for a transition, we use the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + H_{\text{int}})\psi. \quad (10.6)$$

To solve this equation, ψ is expanded in terms of the complete set of unperturbed eigenfunctions, Eq. (10.3):

$$\psi = \sum_n a_n(t) u_n \exp\left(\frac{-iE_n t}{\hbar}\right). \quad (10.7)$$

The coefficients $a_n(t)$ generally depend on time and $|a_n(t)|^2$ is the probability of finding the system at time t in state n with energy E_n . Inserting ψ into the

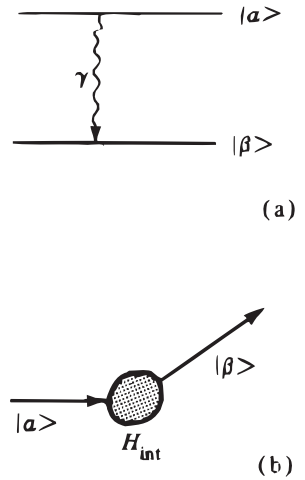


Figure 10.1: The interaction Hamiltonian H_{int} is responsible for transitions from the unperturbed eigenstate $|\alpha\rangle$ to the unperturbed eigenstate $|\beta\rangle$.

Schrödinger equation gives ($\dot{a}_n \equiv da_n/dt$)

$$\begin{aligned} i\hbar \sum_n \dot{a}_n u_n \exp\left(\frac{-iE_n t}{\hbar}\right) + \sum_n E_n a_n u_n \exp\left(\frac{-iE_n t}{\hbar}\right) \\ = \sum_n a_n (H_0 + H_{\text{int}}) u_n \exp\left(\frac{-iE_n t}{\hbar}\right). \end{aligned}$$

With equation (10.4), the second term on the left-hand side and the first term on the right-hand side cancel. Multiplying by u_N^* from the left, integrating over all space, and using the orthonormality relation, produce the result

$$i\hbar \dot{a}_N = \sum_n \langle N | H_{\text{int}} | n \rangle a_n \exp\left[\frac{i(E_N - E_n)t}{\hbar}\right]. \quad (10.8)$$

Here, a convenient abbreviation for the matrix element of H_{int} has been introduced:

$$\langle N | H_{\text{int}} | n \rangle = \int d^3x u_N^*(\mathbf{x}) H_{\text{int}} u_n(\mathbf{x}). \quad (10.9)$$

The set of relations (10.8) for all N is equivalent to the Schrödinger equation (10.6) and no approximation is involved.

A useful approximate solution of Eq. (10.8) is obtained if it is assumed that the interacting system is initially in one particular state of the unperturbed system and if the perturbation H_{int} is weak. In Fig. 10.1, the initial state is $|\alpha\rangle$; it can, for instance, be a well-defined excited level. In terms of the expansion (10.7), the situation is described by

$$a_\alpha(t) = 1, \quad \text{all other } a_n(t) = 0, \quad \text{for } t < t_0. \quad (10.10)$$

Only one of the expansion coefficients is different from zero; all others vanish. The assumption that the perturbation is weak means that, during the time of observation, so few transitions have occurred that the initial state is not appreciably depleted, and other states are not appreciably populated. In lowest order it is then possible to set

$$a_\alpha(t) \approx 1, \quad a_n(t) \ll 1, \quad n \neq \alpha, \quad \text{all } t. \quad (10.11)$$

Equation (10.8) then simplifies to

$$\dot{a}_N = (i\hbar)^{-1} \langle N | H_{\text{int}} | \alpha \rangle \exp\left[\frac{i(E_N - E_\alpha)t}{\hbar}\right].$$

If H_{int} is switched on at the time $t_0 = 0$ and is time-independent thereafter, integration, for $N \neq \alpha$, gives

$$a_N(T) = (i\hbar)^{-1} \langle N | H_{\text{int}} | \alpha \rangle \int_0^T dt \exp\left[\frac{i(E_N - E_\alpha)t}{\hbar}\right]$$

or

$$a_N(T) = \frac{\langle N | H_{\text{int}} | \alpha \rangle}{E_N - E_\alpha} \left\{ 1 - \exp \left[\frac{i(E_N - E_\alpha)T}{\hbar} \right] \right\}. \quad (10.12)$$

The probability of finding the system in the particular state N after time T is given by the absolute square of $a_N(T)$, or

$$P_{N\alpha}(T) = |a_N(T)|^2 = 4 |\langle N | H_{\text{int}} | \alpha \rangle|^2 \frac{\sin^2[(E_N - E_\alpha)T/2\hbar]}{(E_N - E_\alpha)^2}. \quad (10.13)$$

If the energy E_N is different from E_α , then the factor $(E_N - E_\alpha)^{-2}$ depresses the transition probability so much that transitions to the corresponding states can be neglected for large times T . However, there may be a group of states with energies $E_N \approx E_\alpha$, such as shown in Fig. 10.2(a), for which the matrix element $\langle N | H_{\text{int}} | \alpha \rangle$ is almost independent of N . This case occurs, for instance, if the states N lie in the continuum.

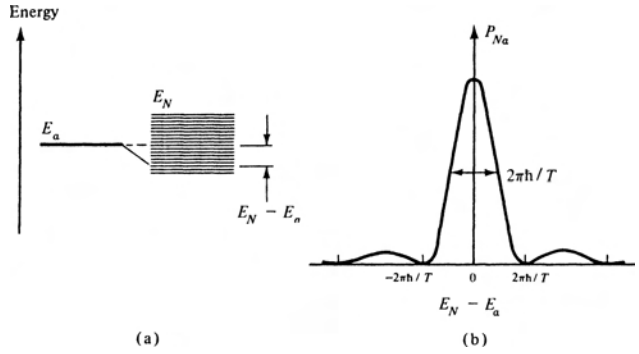


Figure 10.2: (a) Transitions occur mainly to states with energies E_N that are close to the initial energy E_α . (b) Transition probability as a function of the energy difference $E_N - E_\alpha$.

To express the fact that the matrix element is assumed to be independent of N , it is written as $\langle \beta | H_{\text{int}} | \alpha \rangle$. The transition probability is then determined by the factor $\sin^2[(E_N - E_\alpha)T/2\hbar](E_N - E_\alpha)^{-2}$, and it is shown in Fig. 10.2(b). The transition probability is appreciable only within the energy region

$$E_\alpha - \Delta E \text{ to } E_\alpha + \Delta E, \quad \Delta E = \frac{2\pi\hbar}{T}. \quad (10.14)$$

As time increases, the spread becomes smaller: within the limits given by the uncertainty relation, energy conservation is a consequence of the calculation and does not have to be added as a separate assumption.

Equation (10.13) gives the transition probability from one initial state to one final state. The total transition probability to all states E_N within the interval (10.14) is the sum over all individual transitions.

$$P = \sum_N P_{N\alpha} = 4 |\langle \beta | H_{\text{int}} | \alpha \rangle|^2 \sum_N \frac{\sin^2[(E_N - E_\alpha)T/2\hbar]}{(E_N - E_\alpha)^2}, \quad (10.15)$$

where it has been assumed that the matrix element is independent of N . This assumption is good as long as $\Delta E/E_\alpha$ is small compared to 1. With Eq. (10.14), the condition becomes

$$T \gg \frac{2\pi\hbar}{E_\alpha} \approx \frac{4 \times 10^{-21} \text{ MeV-sec}}{E_\alpha(\text{in MeV})}, \quad (10.16)$$

where T is the time of observation. In most experiments, this condition is satisfied.

Now we return to the original problem, shown, for instance, in Fig. 10.1(a). Here, the energy in the initial state is well defined, but in the final state, the emitted photon is free and can have an arbitrary energy (Fig. 10.3). The discrete energy levels E_N of Fig. 10.2(a) consequently are replaced by a continuum. This fact is expressed by writing the energy as $E(N)$. N now labels the energy levels of the photon in the continuum, and it is a continuous variable. The total transition probability follows from Eq. (10.15) if the sum is replaced by an integral, $\sum_N \rightarrow \int dN$:

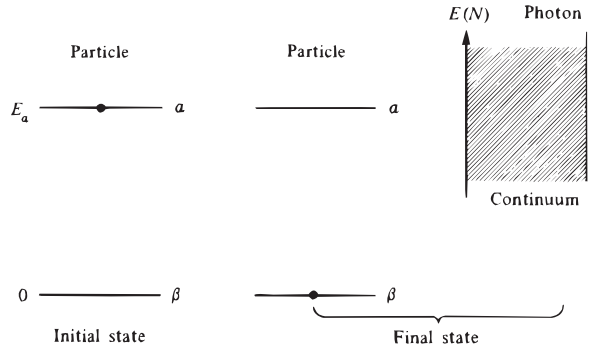


Figure 10.3: In the initial state the subatomic particle is in the excited state α , and no photon is present. In the final state, the subatomic system is in state β , and a photon with energy $E(N)$ has been emitted. The energy of the photon "is in the continuum."

$$P(T) = 4|\langle\beta|H_{\text{int}}|\alpha\rangle|^2 \int \frac{\sin^2[(E(N) - E_\alpha)T/2\hbar]}{(E(N) - E_\alpha)^2} dN. \quad (10.17)$$

The integral extends over the states to which the transitions can occur. Since the integral converges very rapidly, the limits can be extended to $\pm\infty$. With

$$x = \frac{(E(N) - E_\alpha)T}{2\hbar}, \quad dN = \frac{dN}{dE} dE = \frac{2\hbar}{T} \frac{dN}{dE} dx,$$

the transition probability becomes

$$P(T) = 4|\langle\beta|H_{\text{int}}|\alpha\rangle|^2 \frac{dN}{dE} \frac{T}{2\hbar} \int_{-\infty}^{+\infty} dx \frac{\sin^2 x}{x^2}.$$

The integral has the value π , so that the transition probability finally becomes

$$P(T) = \frac{2\pi T}{\hbar} |\langle \beta | H_{\text{int}} | \alpha \rangle|^2 \frac{dN}{dE}. \quad (10.18)$$

The notation $\langle \beta | H_{\text{int}} | \alpha \rangle$ indicates that the transition occurs from states $|\alpha\rangle$ to states $|\beta\rangle$. Since H_{int} is assumed to be time-independent, the transition probability is proportional to the time T . The transition *rate* is the transition probability per unit time, and it is

$$w_{\beta\alpha} = \dot{P}(T) = \frac{2\pi}{\hbar} |\langle \beta | H_{\text{int}} | \alpha \rangle|^2 \frac{dN}{dE}. \quad (10.19)$$

We have thus derived the golden rule. (Actually Fermi called it the *golden rule No. 2*.) It is extremely useful in all discussions of transition processes and we shall refer to it frequently. The factor

$$\frac{dN}{dE} \equiv \rho(E) \quad (10.20)$$

is called the *density-of-states factor*; it gives the number of available states per unit energy, and it will be discussed in Section 10.2.

• In some applications it happens that the matrix element $\langle \beta | H_{\text{int}} | \alpha \rangle$, connecting states of equal energy, vanishes. The approximation that leads to Eq. (10.18) can then be taken one step further. Fermi called this result the *golden rule No. 1*, and it can be stated simply: Replace the matrix element $\langle \beta | H_{\text{int}} | \alpha \rangle$ in Eq. (10.19) by

$$\langle \beta | H_{\text{int}} | \alpha \rangle \longrightarrow - \sum_n \frac{\langle \beta | H_{\text{int}} | n \rangle \langle n | H_{\text{int}} | \alpha \rangle}{E_n - E_\alpha}. \quad (10.21)$$

The one-step transition $|\alpha\rangle \longrightarrow |\beta\rangle$ from the initial to the final state is replaced by a sum over two-step transitions. These proceed from the initial state $|\alpha\rangle$ to all accessible *intermediate* states $|n\rangle$ and from there to the final state $|\beta\rangle$. •

10.2 Phase Space

In the present section, we shall derive an expression for the density-of-states factor $\rho(E) \equiv dN/dE$. We consider first a one-dimensional problem, where a particle moves along the x direction with momentum p_x . Position and momentum of the particles are described simultaneously in an $x - p_x$ plot (phase space). The representation is different in classical and in quantum mechanics. In classical mechanics, position and momentum can be measured simultaneously to arbitrary accuracy, and the state of a particle can be represented by a point (Fig. 10.4(a)). Quantum mechanics, however, limits the description in phase space. The uncertainty relation

$$\Delta x \Delta p_x \geq \hbar$$

states that position and momentum cannot be simultaneously measured to unlimited accuracy. The product of uncertainties must be bigger than $\hbar$, and a particle

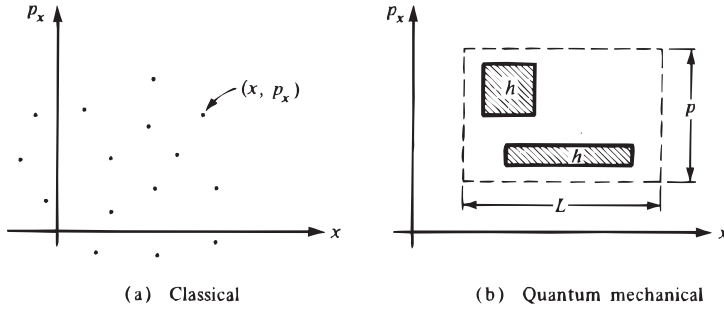


Figure 10.4: Classical and quantum mechanical one-dimensional phase space. In the classical case, the state of a particle can be described by a point. In the quantum case, a state must be described by a cell of volume $h = 2\pi\hbar$.

consequently must be represented by a cell rather than a point in phase space. The shape of the cell depends on the measurements that have been made, but the *volume* is always equal to $h = 2\pi\hbar$. In Fig. 10.4(b), a volume Lp is shown. The maximum number of cells that can be crammed into this volume is given by the total volume divided by the cell volume,

$$N = \frac{Lp}{2\pi\hbar}. \quad (10.22)$$

N is the *number of states* in the volume Lp .⁽¹⁾

The *density of states factor* $\rho(E)$ in one dimension is obtained from Eq. (10.22), with $E = p^2/2m$, as

$$\begin{aligned} \rho(E) &= \frac{dN}{dE} = 2 \frac{dN}{dp} \frac{dp}{dE} = \frac{L}{2\pi\hbar} \frac{2m}{p} \\ &= \frac{L}{2\pi\hbar} \sqrt{\frac{2m}{E}}. \end{aligned} \quad (10.23)$$

The factor 2 in Eq. (10.23) is introduced because for each energy E there are two degenerate states of momentum p and $-p$.

Equation (10.22) can be verified by considering a free wave in a one-dimensional “box” of length L . The normalized solution for the Schrödinger equation in the box,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0 \quad \text{is} \quad \psi = \frac{1}{\sqrt{L}}e^{ikx}.$$

Periodic boundary conditions, $\psi(x) = \psi(x + L)$, give

$$\psi(0) = \psi(L), \quad \text{and} \quad k = \pm \frac{2\pi n}{L}, \quad n = 0, 1, 2, \dots \quad (10.24)$$

¹Note that N is the number of states, not particles. One state can accommodate one fermion but an arbitrary number of bosons.

The number of states per unit momentum interval for $n \gg 1$ is given by

$$\frac{\Delta n}{\Delta p} \approx \frac{dn}{dp} = \frac{1}{\hbar} \frac{dn}{dk} = \frac{L}{2\pi\hbar},$$

in agreement with Eq. (10.22).

Equation (10.22) is valid for a particle with one degree of freedom. For a particle in three dimensions, the volume of a cell is given by $h^3 = (2\pi\hbar)^3$, and the number of states in a volume $\int d^3x d^3p$ in the six-dimensional phase space is

$$N_1 = \frac{1}{(2\pi\hbar)^3} \int d^3x d^3p. \quad (10.25)$$

The subscript 1 indicates that N_1 is the number of states for one particle. If the particle is confined to a spatial volume V , integration over d^3x gives

$$N_1 = \frac{V}{(2\pi\hbar)^3} \int d^3p. \quad (10.26)$$

The density-of-states factor, Eq. (10.20), can now be computed easily:

$$\rho_1 = \frac{dN_1}{dE} = \frac{V}{(2\pi\hbar)^3} \frac{d}{dE} \int d^3p = \frac{V}{(2\pi\hbar)^3} \frac{d}{dE} \int p^2 dp d\Omega, \quad (10.27)$$

where $d\Omega$ is the solid-angle element. With $E^2 = (pc)^2 + (mc^2)^2$, d/dE becomes

$$\frac{d}{dE} = \frac{E}{pc^2} \frac{d}{dp}$$

and consequently (with $(d/dp) \int dp \rightarrow 1$)

$$\rho_1 = \frac{V}{(2\pi\hbar)^3} \frac{pE}{c^2} \int d\Omega. \quad (10.28)$$

For transitions to all final states, regardless of the direction of the momentum $\mathbf{p}$, the density-of-states factor for one particle is

$$\rho_1 = \frac{VpE}{2\pi^2c^2\hbar^3}. \quad (10.29)$$

Next we consider the density of states for *two particles*, 1 and 2. If the total momentum of the two particles is fixed, the momentum of one determines the momentum of the other and the extra degrees of freedom are not really there. The total number of states in momentum space is the same as for one particle, namely N_1 , as in Eq. (10.26). However, the density-of-states factor, ρ_2 , is different from Eq. (10.28) because E is now the total energy of the *two* particles:

$$\rho_2 = \frac{V}{(2\pi\hbar)^3} \frac{d}{dE} \int d^3p_1 = \frac{V}{(2\pi\hbar)^3} \frac{d}{dE} \int p_1^2 dp_1 d\Omega_1, \quad (10.30)$$

where

$$dE = dE_1 + dE_2 = \frac{p_1 c^2}{E_1} dp_1 + \frac{p_2 c^2}{E_2} dp_2.$$

The evaluation is easiest in the c.m. where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, or

$$p_1^2 = p_2^2 \longrightarrow p_1 dp_1 = p_2 dp_2, \quad \text{and} \quad dE = p_1 dp_1 \frac{(E_1 + E_2)}{E_1 E_2} c^2.$$

The density-of-states factor is then given by

$$\rho_2 = \frac{V}{(2\pi\hbar)^3 c^2} \frac{E_1 E_2}{(E_1 + E_2) p_1} \frac{d}{dp_1} \int p_1^2 dp_1 d\Omega_1$$

or

$$\rho_2 = \frac{V}{(2\pi\hbar)^3 c^2} \frac{E_1 E_2 p_1}{(E_1 + E_2)} \int d\Omega_1. \quad (10.31)$$

The extension of Eq. (10.30) to three or more particles is straightforward. Consider three particles; in their c.m. the momenta are constrained by

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0. \quad (10.32)$$

The momenta of two particles can vary independently, but the third one is determined. The number of states therefore is

$$N_3 = \frac{V^2}{(2\pi\hbar)^6} \int d^3 p_1 \int d^3 p_2, \quad (10.33)$$

and the density-of-states factor becomes

$$\rho_3 = \frac{V^2}{(2\pi\hbar)^6} \frac{d}{dE} \int d^3 p_1 \int d^3 p_2. \quad (10.34)$$

For n particles, the generalization of Eq. (10.34) is

$$\rho_n = \frac{V^{n-1}}{(2\pi\hbar)^{3(n-1)}} \frac{d}{dE} \int d^3 p_1 \cdots \int d^3 p_{n-1}. \quad (10.35)$$

We shall encounter an application of Eq. (10.34) in Chapter 11, and we shall discuss the further evaluation there.

10.3 The Classical Electromagnetic Interaction

The energy (Hamiltonian) of a free nonrelativistic particle with mass m and momentum $\mathbf{p}_{\text{free}}$ is given by

$$H_{\text{free}} = \frac{\mathbf{p}_{\text{free}}^2}{2m}. \quad (10.36)$$