that the energy–momentum-conserving δ function implies that

$$q = p'_{\mathrm{b}} - p_{\mathrm{b}} = p_{\mathrm{a}} - p'_{\mathrm{a}}.$$

Can we get a diagrammatic interpretation of our result? We have written it in a way which suggests what the diagram should be. We already know that the $(p_{\mathrm{a}} + p'_{\mathrm{a}})^{\mu}$ and $(p_{\mathrm{b}} + p'_{\mathrm{b}})^{\nu}$ bits are associated with emission or absorption vertices; and we know that in this process there are no free photons in the initial or final states. Therefore the photon must be emitted from one particle and absorbed by the other, as in figure 5.4 (note that there are two ways in which this can happen). In that case, the $-g_{\mu\nu}/q^2$ part must somehow correspond to the *sum* of the two processes shown in figure 5.4! We shall pursue this in more detail in §5.6. For the moment, it is high time that we calculated a cross section.

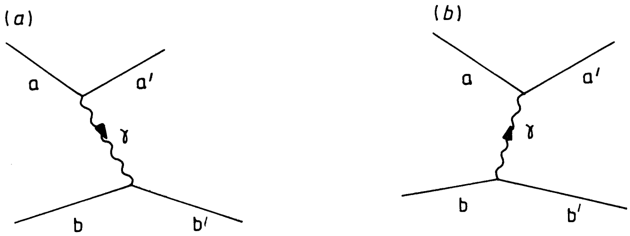

Figure 5.4 The two possible one-photon exchange processes: in (a) particle 'a' emits the photon and particle 'b' absorbs it, while in (b) particle 'b' emits and particle 'a' absorbs.

5.4 Cross section for two-particle scattering

Consider the general two-body process $1 + 2 \to 3 + 4$, where now the numerals label the momenta of the particles. We must first decide on the normalisation of our wavefunctions. We have defined plane wave solutions

$$\phi_{\mathrm{i}} = N_{\mathrm{i}} \mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x} \tag{5.74}$$

so that the corresponding probability density for a positive-energy KG particle is

$$\rho_{\mathrm{i}} = \mathrm{i}\left[\phi_{\mathrm{i}}^{*}\left(\frac{\partial\phi_{\mathrm{i}}}{\partial t}\right) - \left(\frac{\partial\phi_{\mathrm{i}}^{*}}{\partial t}\right)\phi_{\mathrm{i}}\right] = 2N_{\mathrm{i}}^{2}E_{\mathrm{i}}. \tag{5.75}$$

Instead of choosing to normalise our wavefunctions to one particle in a box of volume V, as in NRQM, we shall normalise instead to $2E$ *particles in a volume* V. This is in accord with the fact that ρ_{i} is the time component of a 4-vector and is therefore called a 'covariant normalisation' condition. Either choice is perfectly viable so long as one uses the appropriate flux and phase space factors.

With the condition

$$\int_V \rho_i \mathrm{d}^3 x = 2E_i \tag{5.76}$$

we obtain the result

$$N_i = V^{-1/2}. \tag{5.77}$$

The three steps to derive the cross section may be summarised as follows.

(1) The transition rate per unit volume is defined by

$$P_{\mathrm{fi}} = |\mathscr{A}_{\mathrm{fi}}|^2/VT \tag{5.78}$$

where T is the time of interaction. Either by cavalierly 'squaring the δ function' or rigorously using wavepackets (as discussed, for example, in Chapter 3 of Taylor (1972)), one derives

$$P_{\mathrm{fi}} = (2\pi)^4\delta^4(p_3 + p_4 - p_1 - p_2)(N_1N_2N_3N_4)^2|F|^2 \tag{5.79}$$

where (cf 5.73)

$$\mathscr{A}_{\mathrm{fi}} = -\mathrm{i}(2\pi)^4\delta^4(p_3 + p_4 - p_1 - p_2)N_1N_2N_3N_4\cdot F. \tag{5.80}$$

(2) In order to obtain a quantity which may be compared from experiment to experiment, we must remove the dependence of the transition rate on the incident flux of particles and the number of target particles per unit volume.

(a) The flux of beam particles incident on a stationary target is just the number of particles per unit area which can reach the target in unit time. Thus the 'active' volume is just $|\boldsymbol{v}|$, the velocity of the beam particles, and we have normalised to $2E_1$ particles in a volume V, so the flux factor is

$$|\boldsymbol{v}|\frac{2E_1}{V}. \tag{5.81}$$

(b) The number of target particles per unit volume is just $2E_2/V$ (actually $2m_2/V$ for particle 2 at rest).

To obtain a normalisation-independent quantity, we therefore divide P_{fi} by the flux factor and by the number of target particles per unit volume, i.e. by the factor

$$|\boldsymbol{v}|\frac{2E_1\cdot 2E_2}{V^2}. \tag{5.82}$$

(3) For a physical scattering cross section we must sum over the

available two-particle final states. For one particle in a volume V this amounts to integrating over the usual two-body phase space factor

$$\frac{V}{(2\pi)^3}\mathrm{d}^3p_3\frac{V}{(2\pi)^3}\mathrm{d}^3p_4. \tag{5.83}$$

For our normalisation of $2E$ particles in volume V the available phase space per particle is thus

$$\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{2E_3}\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_4}{2E_4}. \tag{5.84}$$

Putting all this together, the cross section is given by

$$\mathrm{d}\sigma = P_{\mathrm{fi}}\frac{V^2}{2E_1{\cdot}2E_2|\boldsymbol{v}|}\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{2E_3}\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_4}{2E_4}. \tag{5.85}$$

Using our expressions for P_{fi} and N_{i}, the final result is

$$\mathrm{d}\sigma = \frac{|F|^2}{2E_1{\cdot}2E_2|\boldsymbol{v}|}(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\frac{\mathrm{d}^3p_3}{2E_3(2\pi)^3}\frac{\mathrm{d}^3p_4}{2E_4(2\pi)^3}. \tag{5.86}$$

Note that:

(i) the factors involving the normalisation volume V have cancelled;
(ii) we can write the flux factor for collinear collisions in invariant form using the relation (easily verified in a particular frame (see problem 5.4))

$$E_1E_2|\boldsymbol{v}| = [(p_1{\cdot}p_2)^2 - m_1^2m_2^2]^{1/2}; \tag{5.87}$$

(iii) the four-dimensional δ function together with the phase space (including the $(1/2E)$ factors) is sometimes called Lorentz invariant phase space:

$$\mathrm{dLips}(s;\, p_3,\, p_4) \tag{5.88}$$

where the Mandelstam s variable is

$$s = (p_1 + p_2)^2 \tag{5.89}$$

(see e.g. Perkins 1987).

5.5 Explicit evaluation of the a + b → a′ + b′ cross section

We identify $p_{\mathrm{a}} = p_1$, $p'_{\mathrm{a}} = p_3$, $p_{\mathrm{b}} = p_2$ and $p'_{\mathrm{b}} = p_4$ in §5.3, and apply the results of §5.4. The differential cross section is

$$\boxed{\mathrm{d}\sigma = \frac{|F|^2}{4[(p_1{\cdot}p_2)^2 - m_1^2m_2^2]^{1/2}}\mathrm{dLips}(s;\, p_3,\, p_4)} \tag{5.90}$$

where the invariant amplitude F for the electromagnetic interaction is

$$F = (e_a e_b/q^2)(p_1 + p_3)\cdot(p_2 + p_4) \tag{5.91}$$

and the two-particle Lorentz invariant phase space is

$$\boxed{\begin{aligned}\mathrm{dLips}(s; p_3, p_4) &= (2\pi)^4\delta^4(p_3 + p_4 - p_1 - p_2)\\ &\quad\times \frac{1}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{2E_3}\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3p_4}{2E_4}.\end{aligned}} \tag{5.92}$$

To gain familiarity with these factors, we shall evaluate these expressions in the centre-of-momentum (CM) frame defined by

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_3 + \boldsymbol{p}_4 = 0. \tag{5.93}$$

The scattering is described by a CM scattering angle θ_{CM}, as in figure 5.5, and the four 4-momenta are given by

$$\begin{aligned} p_1^\mu &= (E_1, \boldsymbol{p}), & p_3^\mu &= (E_1, \boldsymbol{p}')\\ p_2^\mu &= (E_2, -\boldsymbol{p}), & p_4^\mu &= (E_2, -\boldsymbol{p}')\end{aligned} \tag{5.94}$$

where

$$|\boldsymbol{p}| = |\boldsymbol{p}'| = p \tag{5.95}$$

and

$$W = E_1 + E_2 = E_3 + E_4 = (p^2 + m_1^2)^{1/2} + (p^2 + m_2^2)^{1/2} \tag{5.96}$$

is the CM energy.

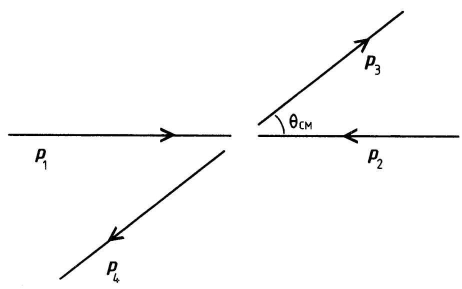

Figure 5.5 Two-body scattering in the CM frame.

5.5.1 Evaluation of two-body phase space in the CM frame

Before we specialise to the CM frame, it is convenient to simplify our expression for dLips:

$$\mathrm{dLips}(s; p_3, p_4) = \frac{1}{(4\pi)^2}\delta^4(p_3 + p_4 - p_1 - p_2)\,\frac{\mathrm{d}^3p_3}{E_3}\,\frac{\mathrm{d}^3p_4}{E_4}. \tag{5.97}$$

Using the 3-momentum δ function, we can eliminate the integral over d^3p_4:

$$\int \frac{\mathrm{d}^3p_4}{E_4}\delta^4(p_3 + p_4 - p_1 - p_2) = \frac{1}{E_4}\delta(E_3 + E_4 - E_1 - E_2). \quad (5.98)$$

On the right-hand side $\boldsymbol{p}_4$ and E_4 are no longer independent variables but are determined by the conditions

$$\boldsymbol{p}_4 = \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3, \qquad E_4 = (|\boldsymbol{p}_4|^2 + m_2^2)^{1/2}. \quad (5.99)$$

Next, convert d^3p_3 to angular variables

$$\mathrm{d}^3p_3 = \mathrm{d}\Omega p_3^2 \mathrm{d}p_3 \quad (5.100)$$

where p_3 now stands for the magnitude of the 3-momentum. The energy and momentum are related by

$$E_3^2 = p_3^2 + m_1^2 \quad (5.101)$$

so that

$$E_3\mathrm{d}E_3 = p_3\mathrm{d}p_3. \quad (5.102)$$

With all these changes we arrive at the result (*valid in any frame*)

$$\mathrm{dLips}(s; p_3, p_4) = \frac{1}{(4\pi)^2}\mathrm{d}\Omega\frac{p_3\mathrm{d}E_3}{E_4}\delta(E_3 + E_4 - E_1 - E_2). \quad (5.103)$$

Now *specialise to the* CM *frame* for which

$$E_3^2 = p^2 + m_1^2 \quad (5.104)$$

$$E_4^2 = p^2 + m_2^2 \quad (5.105)$$

and

$$E_3\mathrm{d}E_3 = p\mathrm{d}p = E_4\mathrm{d}E_4. \quad (5.106)$$

Introduce the variable

$$W' = E_3 + E_4 \quad (5.107)$$

(since $E_3 + E_4$ is only constrained to be equal to $W = E_1 + E_2$ *after* performing the integral over the energy-conserving δ function). Then

$$\mathrm{d}W' = \mathrm{d}E_3 + \mathrm{d}E_4 = \frac{W'}{E_3E_4}p\mathrm{d}p = \frac{W'}{E_4}\mathrm{d}E_3 \quad (5.108)$$

where we have used equation (5.106) in each of the last two steps. Thus the factor

$$p_3\frac{\mathrm{d}E_3}{E_4}\delta(E_3 + E_4 - E_1 - E_2)$$

becomes

$$\frac{p}{W'}\mathrm{d}W'\delta(W' - W) = \frac{p}{W} \tag{5.109}$$

and we arrive at the important result

$$\boxed{\mathrm{dLips}(s;\, p_3, p_4) = \frac{1}{(4\pi)^2}\frac{p}{W}\mathrm{d}\Omega} \tag{5.110}$$

for two-body phase space in the CM frame.

5.5.2 *Flux factor in the CM frame*

Define

$$f^2 = (p_1 \cdot p_2)^2 - m_1^2 m_2^2. \tag{5.111}$$

Using the CM result

$$p_1 \cdot p_2 = E_1 E_2 + p^2 \tag{5.112}$$

a straightforward calculation shows that

$$f = pW. \tag{5.113}$$

We can now write down the CM *differential cross section*

$$\mathrm{d}\sigma = \frac{1}{4f}|F|^2\frac{1}{(4\pi)^2}\frac{p}{W}\mathrm{d}\Omega. \tag{5.114}$$

The result is

$$\boxed{\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{CM}} = \frac{1}{(8\pi W)^2}|F|^2.} \tag{5.115}$$

Before we evaluate $|F|^2$ let us convert this to invariant form as an exercise in changing variables. The 4-momentum transfer squared is defined as the Mandelstam t variable:

$$t = q^2 = (p_1 - p_3)^2 = (p_4 - p_2)^2. \tag{5.116}$$

In the CM frame this reduces to the relation

$$t = -\,2p^2(1 - \cos\theta_{\mathrm{CM}}) \tag{5.117}$$

and hence

$$\mathrm{d}t = 2p^2\mathrm{d}\,(\cos\theta_{\mathrm{CM}}). \tag{5.118}$$

For spinless particles there is cylindrical symmetry about the beam axis

$$\mathrm{d}\Omega_{\mathrm{CM}} = 2\pi\mathrm{d}(\cos\theta_{\mathrm{CM}}) \tag{5.119}$$

and so we arrive at the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\pi}{p^2} \frac{\mathrm{d}}{\mathrm{d}\Omega_{\mathrm{CM}}}. \tag{5.120}$$

Thus the *two-body differential cross section in invariant form* is easily shown to be

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi} \frac{1}{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} |F|^2 \tag{5.121}$$

or, in terms of s,

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi} \frac{1}{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]} |F|^2.} \tag{5.122}$$

These expressions are valid for any unpolarised $2 \to 2$ elastic scattering reaction. For a + b scattering it remains only to calculate $|F|^2$:

$$|F|^2 = \left(\frac{e_{\mathrm{a}} e_{\mathrm{b}}}{q^2}\right)^2 [(p_1 + p_3)\cdot(p_2 + p_4)]^2. \tag{5.123}$$

For elastic scattering the definitions

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{5.124}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{5.125}$$

lead to the relations

$$2p_1 \cdot p_2 = 2p_3 \cdot p_4 = s - m_1^2 - m_2^2 \tag{5.126}$$

$$2p_1 \cdot p_4 = 2p_2 \cdot p_3 = m_1^2 + m_2^2 - u. \tag{5.127}$$

Finally we derive the result for $e_{\mathrm{a}} = e_{\mathrm{b}} = e$:

$$|F|^2 = (4\pi\alpha/t)^2 (s - u)^2 \tag{5.128}$$

where we have explicitly set

$$\alpha = e^2/4\pi \simeq \tfrac{1}{137} \tag{5.129}$$

the fine structure constant in natural units (see Appendices B and C).

It is straightforward to evaluate these invariant expressions in any desired frame. As an example, we consider the case in which 'a' is a particle of charge eZ_{a} and mass m, and 'b' is a particle of charge eZ_{b} and mass M, and we take $m \ll M$. This will then simulate either the electromagnetic scattering of a light particle by a heavy nucleus, say, or a 'spinless' electron by a 'spinless' proton (the complications of spin will occupy us in Chapter 6). We shall work in the laboratory frame

$$p_{\mathrm{b}} = (M, \mathbf{0}) \tag{5.130}$$

in which 'b' is initially at rest. The light particle 4-vectors are

$$p_{\mathrm{a}} = (\omega_{\mathrm{a}}, \boldsymbol{k}) \simeq (|\boldsymbol{k}|, \boldsymbol{k}) \tag{5.131}$$

$$p'_{\mathrm{a}} = (\omega'_{\mathrm{a}}, \boldsymbol{k}') \simeq (|\boldsymbol{k}'|, \boldsymbol{k}') \tag{5.132}$$

neglecting m. Then

$$\begin{aligned} t = q^2 = (p_{\mathrm{a}} - p'_{\mathrm{a}})^2 &= -2|\boldsymbol{k}|\cdot|\boldsymbol{k}'|\cdot(1 - \cos\theta) \\ &= -4|\boldsymbol{k}|\cdot|\boldsymbol{k}'|\cdot \sin^2\theta/2. \end{aligned} \tag{5.133}$$

If we write $p'_{\mathrm{b}} = p_{\mathrm{b}} + (p_{\mathrm{a}} - p'_{\mathrm{a}})$ and square it, we obtain $2p_{\mathrm{b}}\cdot q + q^2 = 0$, from which follows

$$|\boldsymbol{k}|/|\boldsymbol{k}'| = 1 + (2|\boldsymbol{k}|/M)\sin^2\theta/2. \tag{5.134}$$

If we now make the *further* approximation

$$|\boldsymbol{k}| \ll M \tag{5.135}$$

we can set $|\boldsymbol{k}| \simeq |\boldsymbol{k}'|$ and write

$$t \simeq -4|\boldsymbol{k}|^2\sin^2\theta/2 \tag{5.136}$$

$$\mathrm{d}t \simeq 2|\boldsymbol{k}|^2\mathrm{d}(\cos\theta). \tag{5.137}$$

The flux factor in (5.121) is

$$(p_{\mathrm{a}}\cdot p_{\mathrm{b}})^2 - m^2M^2 \simeq M^2\omega_{\mathrm{a}}^2 \simeq M^2|\boldsymbol{k}|^2 \tag{5.138}$$

in the approximation of neglecting m. We also have

$$s = (p_{\mathrm{a}} + p_{\mathrm{b}})^2 = (\omega_{\mathrm{a}} + M)^2 - \boldsymbol{k}^2 \simeq 2|\boldsymbol{k}|M \tag{5.139}$$

and

$$u = (p_{\mathrm{a}} - p'_{\mathrm{b}})^2 \simeq -2|\boldsymbol{k}|M \tag{5.140}$$

so that

$$(s - u)^2 \simeq 16|\boldsymbol{k}|^2M^2. \tag{5.141}$$

Putting the pieces together yields finally

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2 Z_{\mathrm{a}}^2 Z_{\mathrm{b}}^2}{4|\boldsymbol{k}|^2\sin^4\theta/2} \tag{5.142}$$

where $\mathrm{d}\Omega = 2\pi\mathrm{d}(\cos\theta)$. This is recognisable as the *Rutherford scattering cross section.*

We shall meet a similar laboratory frame cross section again in Chapter 6, §6.8, where we shall need to evaluate it without making the low-energy approximation (5.135). This is quite a tricky calculation and is described in Appendix E.