Expanding $(D + A/2)^{5/3}$ and $(-D + A/2)^{5/3}$ in a Taylor series

$$f(x + a) = f(a) + xf'(a) + \frac{x^2}{2}f''(a) + \cdots, \tag{5.32}$$

we obtain

$$\Delta E = \frac{10}{9}C'\frac{(Z - A/2)^2}{A} + \cdots. \tag{5.33}$$

We see that the imbalance between the proton and the neutron number increases the energy of the system (decreasing the binding energy) by the amount specified in (5.33). This justifies the existence of the asymmetry term

$$B_4 = -a_A\frac{(Z - A/2)^2}{A}$$

in the liquid drop model.

Despite its simplicity, the Fermi gas model is able to explain many of the nuclear properties discussed in the previous chapter. At the beginning, the occupation of the states indicates that in a light nucleus $Z \cong N$, since in this way the energy is lowered. For heavy nuclei the Coulomb force makes the proton well shallower than that for neutrons; as a consequence, the proton number is smaller than that of neutrons, being with agreement with the practical occurrence (see figure 5.8b). Another explained characteristic is the verified abundance of even-even nuclei contrasted to the almost nonexistence of stable odd-odd nuclei. It is easy to see why this happens: when we have a nucleon isolated in a level, the lower possible state of energy for a subsequent nucleon is in that same level. In other words, in an odd-odd nucleus we have one isolated proton and one isolated neutron, each in its potential well. But between those states there is, generally, a difference of energy, creating the possibility of passage of one of the nucleons to the well of the other through β-emission and, thus, the nucleus returns to stability.

Finally, we notice that we have described here an independent particle model, and the above results predict the success applying that idea to more sophisticated models, as for instance the shell model that we will study next.

5.4 The Shell Model

This model admits that the nucleons move within the nucleus independently of each other, in the same spirit as the Fermi gas model. The difference is that now the nucleons are not treated as free particles but are subject to a central potential, similar to the central potential that acts on electrons in the atom. At first sight the idea is a bit strange because we cannot, as in the atomic case, identify the origin of a such potential. This difficulty is resolved by assuming that each nucleon moves in an average potential created by the other nucleons, a potential that should be determined in a way to best reproduce the experimental results.

The first proposals of the model appeared at the end of the 1920s, motivated by the fluctuations in the relative abundance and masses of the nuclei along the periodic table.

However, the lack of an apparent theoretical basis and the low acceptance of the idea of independent motion of nucleons, together with poor initial results, meant that the model took a long time to succeed. Finally, the introduction of a spin-orbit term, in 1949, established in a definitive way the shell model as an important tool of vast use in nuclear physics.

We are going to describe the shell model idea in a more formal way. The exact Hamiltonian for a problem of A bodies can be written as

$$H = \sum_{i}^{A} T_i(\mathbf{r}_i) + V(\mathbf{r}_1, \ldots, \mathbf{r}_A), \tag{5.34}$$

where T is the kinetic energy operator and V the potential function.

If we restrict ourselves to two-body interactions (e.g., nucleon-nucleon interaction), (5.34) takes the form

$$H = \sum_{i}^{A} T_i(\mathbf{r}_i) + \frac{1}{2} \sum_{ji} V_{ij}(\mathbf{r}_i, \mathbf{r}_j). \tag{5.35}$$

In the model proposal, the nucleon i feels not the potential $\sum_j V_{ij}$, but a central potential $U(r_i)$, that depends only on the coordinates of nucleon i. This potential can be introduced in (5.35), with the result

$$H = \sum_{i}^{A} T_i(\mathbf{r}_i) + \sum_{i}^{A} U(r_i) + H_{\text{res}}, \tag{5.36}$$

$$H_{\text{res}} = \frac{1}{2} \sum_{ji} V_{ij}(\mathbf{r}_i, \mathbf{r}_j) - \sum_{i} U(r_i). \tag{5.37}$$

H_{res} refers to the *residual interactions*, that is, the part of potential V not embraced by the central potential U. The hope of the shell model is that the contribution of H_{res} is small or, alternatively, that the *shell model Hamiltonian*,

$$H_0 = \sum_{i=1}^{A} \left[T_i(r_i) + U(r_i) \right], \tag{5.38}$$

represents a good approximation for the exact expression of H. Later we shall see that part of the lost accuracy when we pass from (5.35) to (5.38) can be recovered by an approximated treatment of the effect of the residual interaction H_{res}.

The solutions $\Psi_1(\mathbf{r}_1), \Psi_2(\mathbf{r}_2), \ldots$, of the equation

$$H_0 \Psi = E \Psi, \tag{5.39}$$

with respective eigenvalues $E_1, E_2, \ldots$, are called *orbits* or *orbitals*. In the shell model prescription the A nucleons fill the orbitals of lower energy in a way compatible with the Pauli principle. Thus, if the sub-index 1 of Ψ_1, which represents the group of quantum numbers of the orbital 1, includes spin and isospin, we can say that the first nucleon is described by $\Psi_1(\mathbf{r}_1), \ldots,$ and the A-th by $\Psi_A(\mathbf{r}_A)$. Thus, the wavefunction

$$\Psi = \Psi_1(\mathbf{r}_1)\Psi_2(\mathbf{r}_2)\ldots\Psi_A(\mathbf{r}_A) \tag{5.40}$$

is a solution of (5.39) with eigenvalues

$$E = E_1 + E_2 + \cdots + E_A, \tag{5.41}$$

and it would be, in principle, the wavefunction of the nucleus, with energy E given by the shell model. We should have in mind, however, that we are treating a fermion system and that the total wavefunction should be antisymmetric for an exchange of coordinates of two nucleons. Such a wavefunction is obtained from (5.40) for the construction of the *Slater determinant*

$$\Psi = \frac{1}{\sqrt{A!}} \begin{vmatrix} \Psi_1(\mathbf{r}_1) & \Psi_1(\mathbf{r}_2) & \cdots & \Psi_1(\mathbf{r}_A) \\ \Psi_2(\mathbf{r}_1) & \Psi_2(\mathbf{r}_2) & \cdots & \Psi_2(\mathbf{r}_A) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_A(\mathbf{r}_1) & \Psi_A(\mathbf{r}_2) & \cdots & \Psi_A(\mathbf{r}_A) \end{vmatrix}, \tag{5.42}$$

where the change of coordinates (or of the quantum numbers) of two nucleons changes the sign of the determinant.

An inconvenience of the construction in (5.42) is that the function Ψ, by mixing well-defined angular momenta J and isotopic spin T, is no longer an eigenfunction of these operators. The solution to this difficulty involves the construction of linear combinations of Slater determinants that are eigenfunctions of J and T. The problem has a well-known solution, but it involves a great amount of calculation. It is important to make clear, however, that many of the properties of the nuclear states can be extracted from the shell model without knowledge of the wavefunction, as we will see next.

We will analyze what is obtained when one starts with potentials $U(r)$ (5.38) with well-known solutions. Let us initially examine the simple harmonic oscillator. Being a potential that always grows with distance, at first it would seem not to be adaptable to representation of the nuclear potential, which goes to zero when the nucleon is at a larger distance than the radius of the nucleus. It is expected, however, that this is not very important when we analyze just the bound states of the nucleus. The oscillator potential has the form

$$V(r) = \frac{1}{2}m\omega^2 r^2, \tag{5.43}$$

where the frequency ω should be adapted to the mass number A.

We will seek solutions of (5.43) of the type

$$\Psi(\mathbf{r}) = \frac{u(r)}{r} Y_l^m(\theta, \phi), \tag{5.44}$$

where the substitution of Ψ in the Schrödinger equation for a particle reduces the solution of (5.44) to the solution of an equation for u:

$$\frac{d^2u}{dr^2} + \left\{ \frac{2m}{\hbar^2}\left[E - V(r)\right] - \frac{l(l+1)}{r^2} \right\} u = 0. \tag{5.45}$$

The solution of (5.45) with the potential (5.43) is

$$u_{nl}(r) = N_{nl} \exp\left(-\frac{1}{2}\nu r^2\right) r^{l+1} \mathcal{V}_{nl}(r), \tag{5.46}$$

where $\nu = m\omega\hbar$ and $\mathcal{V}_{nl}(r)$ is the associated Laguerre polynomial

$$\mathcal{V}_{nl}(r) = L^{l+\frac{1}{2}}_{n+l-\frac{1}{2}}(\nu r^2) = \sum_{k=0}^{n-1} (-1)^k 2^k \begin{pmatrix} n-1 \\ k \end{pmatrix} \frac{(2l+1)!!}{(2l+2k+1)!!} (\nu r^2)^k, \tag{5.47}$$

where $L_k^\alpha(t)$ are solutions of the equation

$$t\frac{d^2L}{dt^2} + (\alpha + 1 - t)\frac{dL}{dt} + kL = 0. \tag{5.48}$$

From the normalization condition

$$\int_0^\infty u_{nl}^2(r)dr = 1 \tag{5.49}$$

we obtain that

$$N_{nl}^2 = \frac{2^{l-n+3}(2l+2n-1)!!}{\sqrt{\pi}(n-1)![(2l+1)!!]^2} \nu^{l+\frac{3}{2}}. \tag{5.50}$$

The energy eigenvalues corresponding to the wavefunction $\Psi_{nlm}(\mathbf{r})$ are

$$E_{nl} = \hbar\omega\left(2n + l - \frac{1}{2}\right) = \hbar\omega\left(\Lambda + \frac{3}{2}\right) = E_\Lambda, \tag{5.51}$$

where

$$n = 1, 2, 3, \ldots, \quad l = 0, 1, 2, \ldots, \quad \text{and} \quad \Lambda = 2n + l - 2. \tag{5.52}$$

For each value of l there are $2(2l+1)$ states with the same energy (degenerate states). The factor 2 is due to two spin states. However, the eigenvalues that correspond to the same value of $2n + l$ (same value of Λ) are also degenerate. As $2n = \Lambda - l + 2 =$ even, a given value of Λ corresponds to the degenerate eigenstates

$$(n, l) = \left(\frac{\Lambda + 2}{2}, 0\right)\left(\frac{\Lambda}{2}, 2\right), \ldots, (2, \Lambda - 2), (1, \Lambda) \tag{5.53}$$

for Λ even and

$$(n, l) = \left(\frac{\Lambda + 1}{2}, 1\right)\left(\frac{\Lambda - 1}{2}, 3\right), \ldots, (2, \Lambda - 2), (1, \Lambda) \tag{5.54}$$

for Λ odd.

We obtain then that the neutron or proton numbers with eigenvalues E_Λ are given by (we will use $l = 2k$ or $2k + 1$, in the case that Λ is even or odd)

$$N_\Lambda = \sum_{k=0}^{\Lambda/2} 2[2(2k) + 1] \qquad \text{for } \Lambda \text{ even}, \tag{5.55}$$

$$N_\Lambda = \sum_{k=0}^{\Lambda-1/2} 2[2(2k+1) + 1] \qquad \text{for } \Lambda \text{ odd}. \tag{5.56}$$

In both cases, the result is

$$N_\Lambda = (\Lambda + 1)(\Lambda + 2). \tag{5.57}$$

The quantum number Λ defines a *shell* and each shell can accommodate N_Λ protons and N_Λ neutrons.

The accumulated number of particles for all the levels up to Λ is

$$\sum_\Lambda N_\Lambda = \frac{1}{3}(\Lambda + 1)(\Lambda + 2)(\Lambda + 3). \tag{5.58}$$

Based on these results, we can make an estimate of the frequency ω of the harmonic oscillator applied to a nucleus with atomic number A. For a harmonic oscillator, the expectation value of the kinetic energy of a given state is equal to the expectation value of the potential energy. Thus, the sum of the energies of the occupied states in a nucleus of mass A is

$$E = m\omega^2 A\langle r^2 \rangle. \tag{5.59}$$

We can estimate $\langle r^2 \rangle$ using

$$\langle r^2 \rangle \cong \frac{3}{5}R^2, \tag{5.60}$$

with $R \cong 1.2A^{\frac{1}{3}}$. Assuming that $N = Z$ and that all states up to an energy E_Λ are occupied, one gets

$$A = \sum_{\Lambda=0}^{\Lambda_0} 2N_\Lambda = \frac{2}{3}(\Lambda_0 + 1)(\Lambda_0 + 2)(\Lambda_0 + 3) \cong \frac{2}{3}(\Lambda_0 + 2)^2 + \text{terms of order}(\Lambda_0) \tag{5.61}$$

and

$$\frac{E}{\hbar\omega} = \sum_{\Lambda=0}^{\Lambda_0} 2N_\Lambda \left(\Lambda + \frac{3}{2}\right) \cong \frac{1}{2}(\Lambda_0 + 2)^4 - \frac{1}{3}(\Lambda_0 + 2)^3 + \cdots . \tag{5.62}$$

Eliminating $\Lambda_0 + 2$ from the equations above and keeping terms of larger order in $\Lambda_0 + 2$, one gets

$$\frac{E}{\hbar\omega} \cong \frac{1}{2}\left(\frac{3}{2}A\right)^{\frac{4}{3}}. \tag{5.63}$$

Using (5.59) and (5.60) gives

$$\hbar\omega \cong 41A^{-\frac{1}{3}} \text{ MeV}. \tag{5.64}$$

The giant dipole resonances are excitations with $\Delta l = \pm 1$. The position of the peak varies with the mass of the nucleus as $A^{-\frac{1}{3}}$, being a good example of application of (5.64).

The levels predicted by the harmonic oscillator are given in table 5.1. We can observe that the closed shells appear in levels 2, 8, and 20, in agreement with the experimental facts, since the nuclei should close their shells (of protons and of neutrons) with a magic number. But, the same does not happen in closed shells for nucleon numbers larger than 20, which is in disagreement with experience.

Table 5.1 Nucleon distribution for the first shells of a simple harmonic oscillator. The last column indicates the total number of neutrons (or protons) accumulated up to that shell.

$\Lambda = 2n + l - 2$	$E/\hbar\omega$	l	States	N_Λ = number of neutrons (protons)	Total
0	3/2	0	1s	2	2
1	5/2	1	1p	6	8
2	7/2	0,2	2s,1d	12	20
3	9/2	1,3	2p,1f	20	40
4	11/2	0,2,4	3s,2d,1g	30	70
5	13/2	1,3,5	3p,2f,1h	42	112
6	15/2	0,2,4,6	4s,3d,2g,1i	56	168

For an infinite square well we have the same approximate situation. The solutions for that potential obey the equation

$$\frac{d^2u}{dr^2} + \left[\frac{2m}{\hbar^2}E - \frac{l(l+1)}{r^2}\right]u = 0, \tag{5.65}$$

whose solutions

$$u = A\, r\, j_l(kr) \tag{5.66}$$

involve spherical Bessel functions that obey the boundary condition

$$j_l(kR) = 0, \tag{5.67}$$

where R is the radius of the nucleus and $k = \sqrt{2mE}/\hbar$. From (5.66) and (5.67) we build the allowed states for that potential. Table 5.2 shows these states, the nucleon numbers admitted in each of them, and the nucleon numbers that close the shells. Again here the magic numbers are only reproduced in the initial shells.

The nuclear potential should actually have an intermediary form between the harmonic oscillator and the square well, not being as smooth as the first or as abrupt as the second. It is common to use the "Woods-Saxon" form $V = V_0/\{1 + \exp[(r - R)/a]\}$, where V_0, r, and a are adjustable parameters. A numerical solution of the Schrödinger equation with such a potential does not supply us, however, with the expected results.

A considerable improvement was obtained in 1949 by Maria Mayer [Ma49] and, independently, by Haxel, Jensen, and Suess [Ha49], with the introduction of a term of spin-orbit interaction in the form

$$f(r)\mathbf{l} \cdot \mathbf{s}, \tag{5.68}$$

Table 5.2 Proton (or neutron) distribution for the first shells of an infinite square well. The principal quantum number *n* indicates the order in that a zero appears for a given *l* in eq. (5.67). Notice that here there is no longer the degeneracy in *l*. The third column gives the proton and neutron numbers that can be fitted in each orbit.

orbit: nl	kR	$2(2l+1)$	Total
1s	3.142	2	2
1p	4.493	6	8
1d	5.763	10	18
2s	6.283	2	20
1f	6.988	14	34
2p	7.725	6	40
1g	8.183	18	58
2d	9.095	10	68
3s	9.425	2	70

where $f(r)$ is a radial function that should be obtained by comparison with experiments. However, we will soon see that its form is not important for the effect that we want.

A spin-orbit term already appears in atomic physics as a result of the interaction between the magnetic moment of the electrons and the magnetic field created by orbital motion. In nuclear physics this term has a different nature and is related to the quantum field properties of an assembly of nucleons.

We will see that the addition of such a term to the potential of (5.43) alters the energy values. The new values are given to first order by

$$E = \int \Psi^* H \Psi = \left(n + \frac{3}{2}\right)\hbar\omega + \alpha \int \Psi^* f(r) \mathbf{l} \cdot \mathbf{s}\, \Psi, \tag{5.69}$$

where α is a proportionality constant. If we suppose now that the spin-orbit term is small and that it can be treated as a small perturbation, the wavefunctions in (5.69) are basically those of a central potential. Recalling that $\mathbf{l} \cdot \mathbf{s} = (j^2 - l^2 - s^2)/2$, we have

$$\begin{aligned} \int \Psi^* \mathbf{l} \cdot \mathbf{s}\, \Psi &= \frac{l}{2} \qquad \text{for } j = l + \frac{1}{2}, \\ \int \Psi^* \mathbf{l} \cdot \mathbf{s}\, \Psi &= -\frac{1}{2}(l+1) \qquad \text{for } j = l - \frac{1}{2}. \end{aligned} \tag{5.70}$$

Thus, the spin-orbit interaction removes the degeneracy in j and, anticipating that the best experimental result will be obtained if the orbitals for larger j have the energy lowered, we admit a negative value for α, allowing us to write for the energy increment

$$\Delta E\mid_{j=l+1/2} = -\mid\alpha\mid\langle f(r)\rangle\frac{l}{2}, \tag{5.71}$$

$$\Delta E\mid_{j=l-1/2} = +\mid\alpha\mid\langle f(r)\rangle\frac{1}{2}(l+1). \tag{5.72}$$

Figure 5.9 exhibits the level scheme of a central potential with the introduction of the spin-orbit interaction. It is easy to see the effect of (5.71) and (5.72) in the energy distribution of the levels. Referring now to a shell as a group of levels of closed energy, not necessarily associated to only one principal quantum number of the oscillator, we obtain a perfect description of all the magic numbers.

We will use this level scheme[1] to establish what one refers to as the *single particle model* or *extreme shell model*. In this version, the model allows that an odd nucleus is composed of an inert even-even core plus an unpaired nucleon, and that this last nucleon determines the properties of the nucleus. This idea was discussed earlier; what we can now do is to determine, starting from figure 5.9, in which state we find the unpaired nucleon.

Let us take ^{17}O as an example. This nucleus has a shell closed with 8 protons but has a remaining neutron above the closed core of 8 neutrons. A quick examination of figure 5.9 indicates that this neutron finds itself in level $1d_{5/2}$. We can say that the neutron *configuration* of this nucleus is

$$(1s_{1/2})^2\,(1p_{3/2})^4\,(1p_{1/2})^2\,(1s_{1/2})^2\,(1d_{5/2})^1,$$

it being evident that in this definition we list the filled levels for the neutrons, with the upper indices, equal to $2j+1$, indicating the number of particles in each of them. It is also common to restrict the configuration to the partially filled levels, the subshells being completely ignored. Thus, the configuration of neutrons in ^{17}O would be $(1d_{5/2})^1$. The prediction of the model is, in this case, that the spin of the ground state of ^{17}O is $\frac{5}{2}$ and the parity is positive ($l=2$). This prediction is in agreement with experiments. For similar reasons, the model predicts that the ground state of ^{17}F is a $\frac{5}{2}^+$ state and this is indeed the measured value.

The extreme shell model works well when we have a nucleon above a closed shell, as in the examples above. It also works well for a hole (absence of a nucleon) in a closed shell. Examples of this case are the nuclei ^{15}O and ^{15}N for which the model predicts correctly a $\frac{1}{2}^-$ ground state. There are situations, however, in which the model needs a certain adaptation. Such is the case, for example, for the stable nuclei ^{203}Tl and ^{205}Tl. They have 81 protons, with a resulting hole in $\frac{11}{2}$. Their ground state is, however, $\frac{1}{2}^+$ instead of $\frac{11}{2}^-$. In order to understand what happens it is necessary to recall that the model, in the simple form we are using, totally neglects the individual interactions of the nucleons; a correction of the model would be to take into account certain nucleon-nucleon interactions that we know are present and that are part of the residual interaction (5.37).

[1] Each level of figure 5.9, characterized by the quantum numbers n, l, j, contains $2j+1$ nucleons of a same type and is also referred to as a subshell.

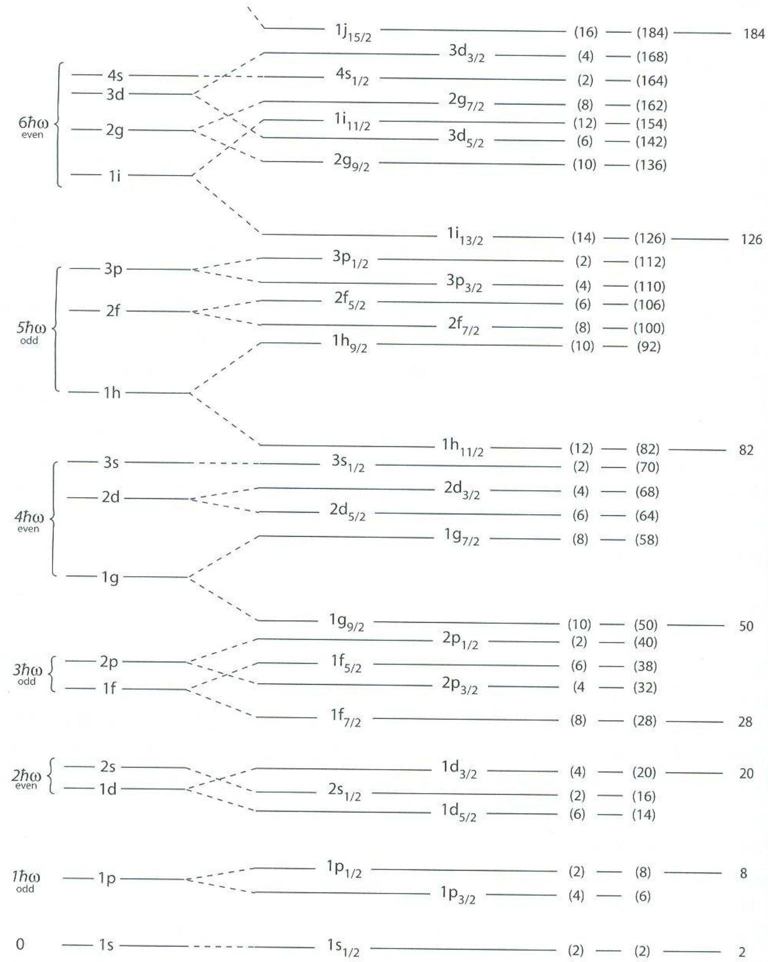

Figure 5.9 Level scheme of the shell model showing the break of the degeneracy in j caused by the spin-orbit interaction term and the emergence of the magic numbers in the shell closing. The values in the first set of parentheses indicate the number of nucleons of each type that the level admits and the values in the second set of parentheses provide the total number of nucleons of each type up to that level. Finally, the numbers outside parentheses indicate the total number of nucleons at shell-closure, reobtaining the magic numbers in their entirety. The ordering of the levels is not rigid, and there could be level inversions when changes occur in the form of the potential [MJ55].

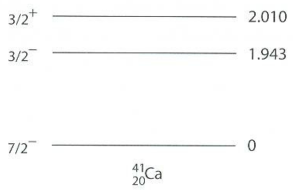

Figure 5.10 Level scheme for ^{41}Ca. The values to the right are the energies in MeV.

A class of interactions of special interest is the one that involves a proton (or a neutron) pair of equal orbits n, l, and j with symmetric values of m_j. A collision of these particles can take the pair to other orbits with the same quantum numbers n, l, j but with new projections m_j' and $-m_j'$. These collisions conserve energy (the $2j+1$ states are degenerate), angular momentum, and parity, and we expect that there is a permanent alternation between the several possible values of the pair $m_j, -m_j$. The interaction between two nucleons in these circumstances is commonly called the *pairing force*. It leads to an increase of the binding energy of the nucleus; since the nucleons belong to the same orbit, their wavefunctions have the same space distribution and the average proximity between them is maximum. As the nuclear force is attractive, this leads to an increase in the binding energy. The pairing force is responsible for the pairing term (5.6) of the mass formula (5.7). It is the same type of force that, acting between the conduction electrons of a metal, in special circumstances and at low temperatures, yields the superconductivity phenomenon [Ba57].

The pairing force increases with the value of j, since the larger the angular momentum, the larger the location of the wavefunction of the nucleon around a classical orbit, and also the stronger the argument of the previous paragraph. This implies that it is sometimes more energetically advantageous that the isolated nucleon is not in the last level but below it, leaving the last level for a group of paired nucleons. This happens in our example of Tl; the hole is located not in $h_{\frac{11}{2}}$ but in $s_{\frac{1}{2}}$, leaving the orbital $h_{\frac{11}{2}}$ (of high j) occupied by a pair of protons. Another example is ^{207}Pb, for which the hole in the closed shell of 126 neutrons is not in $i_{\frac{13}{2}}$ but in $p_{\frac{1}{2}}$, resulting in the value $\frac{1}{2}^-$ for the spin of its ground state.

An example of another kind is ^{23}Na. This nucleus has the last 3 protons in the orbital $1d_{\frac{5}{2}}$. The value of its spin is, however, $\frac{3}{2}$. This is an example of a flaw in the predictions of the extreme shell model. Here, it is the coupling between the three nucleons that determines the value of the spin and not separately the value of j of each of them. This type of behavior will be analyzed in the following section.

Having established the outline of operation of the shell model, it is easy to apply it to the determination of the excited states of nuclei. In the case of ^{41}Ca (figure 5.10), the ground state is $\frac{7}{2}^-$, since the extra neutron occupies the orbital $f_{\frac{7}{2}}$. The first excited level corresponds to a jump of that neutron to $p_{\frac{3}{2}}$, generating the state $\frac{3}{2}^-$. The second excited