$$\frac{1}{2}\left[\frac{\mathrm{d}r}{\mathrm{d}t}\right]^2 = \frac{Gm_0}{r} - \frac{Gm_0}{r_0}.$$

It follows that the time for free fall to the centre of the sphere is given by

$$t_{FF} = \int_{r_0}^{0} \frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r = -\int_{r_0}^{0}\left[\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right]^{-1/2} \mathrm{d}r.$$

This may be simplified by introducing the parameter $x = r/r_0$ to give

$$t_{FF} = \left[\frac{r_0^3}{2Gm_0}\right]^{1/2} \int_0^1 \left[\frac{x}{1-x}\right]^{1/2} \mathrm{d}x.$$

The integral in this equation may be evaluated by substitution of $x = \sin^2\theta$ to give $\pi/2$.

We have shown that the free-fall time for a shell of radius r_0 enclosing mass m_0 depends on m_0/r_0^3, i.e. it is determined by the average density of the matter enclosed. It follows that, in the absence of an internal pressure gradient, a sphere with an initial, uniform density of ρ will collapse as a whole in a time given by

$$t_{FF} = \left[\frac{3\pi}{32G\rho}\right]^{1/2}. \tag{1.4}$$

Collapse under gravity is never completely unopposed. In practice the energy released by the gravitational field of the collapsing system is usually dissipated into random thermal motion of the constituents, thereby creating a pressure which opposes further collapse. However, free fall is a relevant approximation if energy is easily lost by radiation, or if the constituents of the collapsing system can absorb energy by excitation or dissociation. For example, an interstellar cloud of molecular hydrogen can collapse rapidly as long as it is transparent to its own radiation, or as long as hydrogen molecules can be dissociated into atomic hydrogen, or as long as atomic hydrogen can be ionized. But the gravitational energy released in an opaque cloud of ionized hydrogen will be trapped as internal thermal motion. The internal pressure will rise and slow down the rate of collapse. The cloud will then approach hydrostatic equilibrium.

Hydrostatic equilibrium

Figure 1.1 and Eq. (1.3) indicate that an element of matter at a distance r from the centre of a spherical system will be in hydrostatic equilibrium if the pressure gradient at r is

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}. \tag{1.5}$$

The whole system is in equilibrium if this equation is valid at all radii, r. In this case it is possible to derive a simple relation between the average internal pressure and the gravitational potential energy of the system.

To derive this relation we multiply Eq. (1.5) by $4\pi r^3$ and integrate from $r = 0$ to $r = R$ to obtain

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = -\int_0^R \frac{Gm(r)\rho(r)4\pi r^2}{r} dr.$$

Both sides of this equation have simple physical significance. The right-hand side is simply the gravitational potential energy of the system:

$$E_{GR} = -\int_{m=0}^{m=M} \frac{Gm(r)}{r} dm, \tag{1.6}$$

where dm is the mass between r and $r + dr$; i.e. $\rho(r)\ 4\pi r^2\ dr$. The left-hand side can be integrated by parts to give

$$[P(r)4\pi r^3]_0^R - 3\int_0^R P(r)4\pi r^2\, dr.$$

The first term is zero because the pressure on the outside surface at $r = R$ is zero. The second term is equal to $-3\langle P \rangle V$, where V is the volume of the system and $\langle P \rangle$ is the volume-averaged pressure. Hence we conclude that the average pressure needed to support a system with gravitational energy E_{GR} and volume V is given by

$$\langle P \rangle = -\frac{1}{3}\frac{E_{GR}}{V}. \tag{1.7}$$

In words, the average pressure is one-third of the density of the stored gravitational energy. This expression for the average pressure needed to support a self-gravitating system is called the *virial theorem*.

The physical origin of this pressure depends on the system. In Chapter 2 we shall consider the pressures generated by classical and quantum gases of both non-relativistic and ultra-relativistic particles. But at this stage we would like to focus on the relation between the pressure and the internal energy density due to the translational motion of the particles, and in so doing we shall emphasize the profound difference in the behaviour of non-relativistic and ultra-relativistic systems.

To derive this relation we consider a gas of N particles in a cubical box of volume L^3 with its edges orientated along the x, y and z axes. Initially we ultra-relativistic: according to Eq. (1.13) the binding energy becomes small, and small changes in the total energy are accompanied by large changes in the internal and gravitational energies. These considerations suggest that stars with a mass greater than 50–100 solar masses are easily disrupted. And indeed such stars are rare.

1.4 THE SUN

As our nearest star, the sun has a special role as a source of precise astrophysical information. For example, we know its mass, radius, geometric shape and age, and also the luminosity and spectrum of electromagnetic radiation from its surface. This observational information is used in theoretical models of the sun to predict the physical characteristics of the solar interior. The most detailed model of the sun is the Standard Solar Model, which is described by Bahcall (1989). Some of the input parameters for this model and some of the calculated solar properties are listed in Table 1.2.

Our aim in this section is to consider the sun in its simplest terms in order to illustrate basic astrophysical concepts and to fix the order of magnitude of astrophysical quantities.

TABLE 1.2 The main physical properties of the sun[a]

Property	Value
Mass	$M_{\odot} = 1.99 \times 10^{30}$ kg
Radius	$R_{\odot} = 6.96 \times 10^{8}$ m
Photon luminosity	$L_{\odot} = 3.86 \times 10^{26}$ W
Effective surface temperature	$T_E = 5780$ K
Age	$t_{\odot} \approx 4.55 \times 10^{9}$ years
Central density	$\rho_c = 1.48 \times 10^{5}$ kg m^{-3}
Central temperature	$T_c = 15.6 \times 10^{6}$ K
Central pressure	$P_c = 2.29 \times 10^{16}$ Pa

[a] The measured properties are the mass, radius, photon luminosity, and surface temperature. The estimate for the age is largely based on geological studies. The properties at the centre of the sun are calculated with the aid of the Standard Solar Model; see Bahcall (1989) for more detail.

Pressure, density and temperature

The sun is a star of mass $M_{\odot} \approx 2 \times 10^{30}$ kg. The gravitational contraction of the sun was halted about 5 billion years ago by the ignition of 'hydrogen burning', i.e. the thermonuclear fusion of hydrogen to form helium. During its current hydrogen burning phase the solar radius is $R_{\odot} \approx 7 \times 10^{8}$ m and the average density $\langle \rho \rangle$ is 1.4×10^{3} kg m^{-3}. The time for free fall under gravity for an object of this density is given by Eq. (1.4),

$$t_{FF} = \left[\frac{3\pi}{32G\langle\rho\rangle}\right]^{1/2} \approx \frac{1}{2} \text{ hour.}$$

As this time bears no relation to the sun we observe, we safely conclude that the sun is not in free fall and that the internal pressure gradient within the sun must play an essential role in opposing gravity. Indeed, as there is no evidence for major changes in the sun during the geological lifetime of the earth, we can conclude that the sun has been close to hydrostatic equilibrium for at least 4.5 billion years. Hydrostatic equilibrium implies we can use the virial theorem to find the average pressure supporting the sun; using Eqs. (1.7) and (1.16) we find

$$\langle P\rangle = -\frac{1}{3}\frac{E_{GR}}{V} \approx \frac{GM_\odot^2}{4\pi R_\odot^4} \approx 10^{14} \text{ Pa.} \tag{1.29}$$

Hence the interior of the sun provides an environment in which matter and radiation interact at high temperature such that, on average, the pressure is about a billion times atmospheric pressure and the density is comparable with normal water. The thermal physics needed to understand matter and radiation under these extreme conditions will be reviewed in Chapter 2; the ionization of gases and the equations of state for non-relativistic, ultra-relativistic, classical and quantum gases will be discussed. This discussion indicates that we are justified in making the simple and bold assumption that the sun is primarily supported by the pressure of an ideal classical gas of electrons and ions. Thus, the average pressure inside the sun is given by

$$\langle P\rangle = \frac{\langle\rho\rangle}{\overline{m}} kT_I, \tag{1.30}$$

where T_I is the typical internal temperature and $\overline{m}$ is the average mass of the gas particles. For ionized hydrogen $\overline{m} = 0.5$ amu, the average mass of a proton and an electron. In fact, the Standard Solar Model assumes that the sun was formed from material which was 71% hydrogen, 27% helium and 2% of heavy elements, such as carbon, oxygen and iron. When fully ionized this yields an average gas particle mass of $\overline{m} \approx 0.61$ amu.

It is easy to combine Eqs. (1.29) and (1.30) to estimate the typical temperature inside the sun. We obtain

$$kT_I \approx \frac{GM_\odot \overline{m}}{3R_\odot} \approx 0.5 \text{ keV} \quad \text{or} \quad T_I \approx 6 \times 10^6 \text{ K.} \tag{1.31}$$

Of course the actual temperature inside the sun, like the density and the pressure, increases towards the centre. The central temperature, density and pressure given by the Standard Solar Model are listed in Table 1.2.