Table 3.1 Isospin wavefunction for the two-nucleon system.

| Isospin wavefunction | $T$ | $T_z$ | Symmetry by isospin exchange |
|---|---|---|---|
| $\phi_1^1 = \pi(1)\pi(2)$ | 1 | 1 | Triplet (symmetric) |
| $\phi_1^0 = \frac{1}{\sqrt{2}}[\pi(1)\nu(2) + \pi(2)\nu(1)]$ | 1 | 0 | |
| $\phi_1^{-1} = \nu(1)\nu(2)$ | 1 | −1 | |
| $\phi_0^0 = \frac{1}{\sqrt{2}}[\pi(1)\nu(2) - \pi(2)\nu(1)]$ | 0 | 0 | Singlet (antisymmetric) |

We denote $\pi$ as a state of the proton and $\nu$ as a state of the neutron, so that $\pi(1)\,\nu(2)$ means that the first nucleon is a proton and the second is a neutron. We can build the isospin part $\phi_{T_z}^T$ of the wavefunction of the two-nucleon system in a similar way to the case of spin, as indicated in table 3.1.

## 3.2 Phenomenological Potentials

In the phenomenological method one uses the appropriate functional form for the potential with a sufficient amount of parameters. The parameters are chosen so that the potential describes as closely as possible the experimental data of the NN system. There are two classes of such potentials: *local* and *nonlocal* potentials.

## 3.3 Local Potentials

The following general ansatz is made for the potential as a function of the relevant degrees of freedom of both nucleons:

$$V(1,2) = V(\mathbf{r}_j, \mathbf{p}_j, \boldsymbol{\sigma}_j, \boldsymbol{\sigma}_j;\ j = 1,2). \tag{3.2}$$

Symmetry and invariance properties of the Hamiltonian operator limit the general form of the interaction (see Appendix C). These properties are the requirement of invariance through translation, rotation, Galilean transformations, and particle exchange in connection with the Pauli principle, that is,

$$V(1,2) = V(2,1). \tag{3.3}$$

To account for these invariance properties one introduces the relative and center-of-mass coordinates and momenta, where the small mass difference between the neutron

**Table 3.2 Tensors in two-nucleon space.**

| Type | Operator | Parity | Time reversal | Number |
|---|---|---|---|---|
| scalar | 1 | + | + | 1 |
| scalar | $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ | + | + | 1 |
| vector | $\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2$ | + | + | 3 |
| vector | $\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$ | + | − | 3 |
| vector | $\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$ | + | − | 3 |
| tensor | $\left[\boldsymbol{\sigma}_1^{[1]} \times \boldsymbol{\sigma}_2^{[1]}\right]^{[2]}$ | + | + | 5 |
| | Total number | | | 16 |

and the proton is neglected:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \qquad \mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2),$$
$$\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2), \qquad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2. \tag{3.4}$$

The requirement of invariance under *translations* $\mathbf{r} \to \mathbf{r}_1 + \mathbf{a}$ leads to the condition

$$V(\mathbf{r}, \mathbf{p}, \mathbf{R}, \mathbf{P}) = V(\mathbf{r}, \mathbf{p}, \mathbf{R} + \mathbf{a}, \mathbf{P}),$$

and invariance under *Galilean transformation* $\mathbf{p}_j \to \mathbf{p}_j + \mathbf{p}_0$ implies

$$V(\mathbf{r}, \mathbf{p}, \mathbf{R}, \mathbf{P}) = V(\mathbf{r}, \mathbf{p}, \mathbf{R}, \mathbf{P} + 2\mathbf{p}_0).$$

Since $\mathbf{a}$ and $\mathbf{p}_0$ can take any values, these relations mean that $V$ cannot depend on $\mathbf{R}$ and $\mathbf{P}$, so it can only possess the form

$$V(1,2) = V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_j, \boldsymbol{\tau}_j;\ j = 1, 2). \tag{3.5}$$

Next we study the *rotational invariance* property. This determines the structure of the spin degrees of freedom. Any function $f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ represents a $4 \times 4$ matrix in the space of two-nucleon spin that can be spanned by a linear combination of 16 matrices. These can be classified by their tensor properties, as shown in table 3.2. The indices [1] and [2] refer the coupling scheme of two tensor operators $T_1^{[L_1]}$ and $T_2^{[L_2]}$ into a new operator

$$T_{[M]}^{[L]} = \left[T_1^{[L_1]} \times T_2^{[L_2]}\right]_{[M]}^{[L]} = \sum_{M_1 M_2} (L_1 M_1 L_2 M_2 | LM) T_{[M_1]}^{[L_1]} T_{[M_2]}^{[L_2]}.$$

The vector operator $\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2$ in the third row of table 3.2 does not carry the similar notation, $\left[\boldsymbol{\sigma}_1^{[1]} \times \boldsymbol{\sigma}_2^{[1]}\right]^{[1]}$, for the sake of simplicity. For more details on tensor operators, see Appendix B.

Table 3.3 Tensors build from r and p.

| Type | Operator | Parity | Time reversal |
|---|---|---|---|
| scalar | $\mathbf{r}^2$ | + | + |
| scalar | $\mathbf{p}^2$ | + | + |
| scalar | $\mathbf{r} \cdot \mathbf{p}$ | + | − |
| vector | $\mathbf{r}$ | − | + |
| vector | $\mathbf{p}$ | − | − |
| vector | $\mathbf{r} \times \mathbf{p}$ | + | − |
| tensor | $[\mathbf{r}^{[1]} \times \mathbf{r}^{[1]}]^{[2]}$ | + | + |
| tensor | $[\mathbf{p}^{[1]} \times \mathbf{p}^{[1]}]^{[2]}$ | + | + |
| tensor | $[\mathbf{r}^{[1]} \times \mathbf{p}^{[1]}]^{[2]}$ | + | − |

When one constructs the potentials in terms of these linear combinations, one must be sure that the result is a scalar and that the symmetries under particle exchange, parity, and time reversal are observed. This means that one has to combine the vector and tensor symmetry operators with the corresponding vector and tensor operators obtained from $\mathbf{r}$ and $\mathbf{p}$. The possible operators obtained in this way are shown in table 3.3.

Due to consideration of symmetry and invariance properties, only the following vector-vector and tensor-tensor combinations are possible:

(a) Vector-vector: spin-orbit operator

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}(\mathbf{r} \times \mathbf{p}) \cdot (\sigma_1 + \sigma_2). \tag{3.6}$$

(b) Tensor-tensor:

$$\begin{aligned}
&[\mathbf{r}^{[1]} \times \mathbf{r}^{[1]}]^{[2]} \cdot \left[\sigma_1^{[1]} \times \sigma_2^{[1]}\right]^{[2]} = (\sigma_1 \cdot \mathbf{r})(\sigma_2 \cdot \mathbf{r}) - \frac{1}{3}\sigma_1 \cdot \sigma_2\, r^2, \\
&[\mathbf{p}^{[1]} \times \mathbf{p}^{[1]}]^{[2]} \cdot \left[\sigma_1^{[1]} \times \sigma_2^{[1]}\right]^{[2]} = (\sigma_1 \cdot \mathbf{p})(\sigma_2 \cdot \mathbf{p}) - \frac{1}{3}\sigma_1 \cdot \sigma_2\, p^2, \\
&\left[\sigma_1^{[1]} \times \sigma_2^{[1]}\right]^{[2]} \cdot [\mathbf{r}^{[1]} \times \mathbf{p}^{[1]}]^{[2]}\ \mathbf{r} \cdot \mathbf{p}.
\end{aligned} \tag{3.7}$$

Instead of the last tensor operator, one uses the equivalent square of the spin-orbit operator $(\mathbf{L} \cdot \mathbf{S})^2$.

From the above considerations we see that the nucleon-nucleon potential, which respects invariance under particle exchange, translation, Galilean transformation, rotation, parity,

and time reversal, is given by

$$V(1,2) = V_C + V_S(\sigma_1 \cdot \sigma_2) + V_T S_{12}(\mathbf{r}) + V'_T S_{12}(\mathbf{p}) + V_{LS}\mathbf{L}\cdot\mathbf{S} + V_Q(\mathbf{L}\cdot\mathbf{S})^2, \tag{3.8}$$

where the operator $S_{12}$ is given by

$$S_{12} = 3\left(\sigma_1 \cdot \frac{\mathbf{r}}{r}\right)\left(\sigma_2 \cdot \frac{\mathbf{r}}{r}\right) - (\sigma_1 \cdot \sigma_2). \tag{3.9}$$

In (3.8) the quantities $V_\alpha$ with $\alpha \in \{C, S, T, T', LS; Q\}$ are scalar functions of the remaining scalars $r^2$, $p^2$, and $(\mathbf{r}\cdot\mathbf{p})^2$. Due to the relation

$$(\mathbf{r}\cdot\mathbf{p})^2 = r^2p^2 - L^2, \tag{3.10}$$

one chooses instead the variables $r^2$, $p^2$, and $L^2$ as the independent ones. One must also be sure that the total $V$ is a Hermitian operator.

As a last point, we have to consider the isospin dependence of the interaction. The experimental data indicate that the NN interaction is approximately independent of the charge state of the nucleons, that is, of nn, pp, or np. In fact, the states $\phi_1^1$ (di-proton) and $\phi_1^{-1}$ (di-neutron) discussed in the last chapter constitute, together with $\phi_1^0$, a triplet in the isospin space. Now we want to know if some member of that triplet can be part of a bound state of the two particles. To show that this is not possible, let us examine the ground state of the deuteron. We saw that this state has $J = 1$, $S = 1$, and $l = 0$. The last value indicates that the space part is symmetrical and that $S = 1$ also corresponds to a symmetrical spin part. As $\Psi$ in (3.1) should be antisymmetric, $\phi_T^{T_z}$ should also be antisymmetric for the ground state of the deuteron, and the isospin wavefunction of that state can only be $\phi_0^0$. The function $\phi_1^0$ is, therefore, the isospin wavefunction of an excited state of the deuteron. But we know experimentally that this state is not bound. As the nuclear force does not depend on the charge, the absence of a bound state for $\phi_1^0$ should be extended to $\phi_1^1$ and $\phi_1^{-1}$. This last result exhibits that *the bound proton-proton or neutron-neutron system does not exist.*

But, how can states with $T = 1$ and $T = 0$ correspond to different energies if the nuclear forces are independent of the charge (isospin)? This is due to the dependence of the nuclear force on the spin. To each group of isospin states is associated a different orientation for the spins, so that to each group correspond different energies. The dependence of the nuclear force on the spin has a connection with the fact that there is no state for the deuteron other than the ground state (triplet spin). The force between the proton and the neutron when they have antiparallel spins (singlet) is smaller than when they have parallel spins (triplet), not strong enough to form a bound state. This force has a value just a little below that necessary to produce a bound state.

One can formally account for isospin independence by using the commutator property $[H, T_\pm] = 0$, where $\mathbf{T}$ is the total isospin operator $\mathbf{T} = \mathbf{t}_1 + \mathbf{t}_2$. Together with the charge conservation property $[H, T_z] = 0$, it follows that $[H, \mathbf{T}^2] = 0$, that is, an invariance under complete rotations in the isospin space. In other words, the interaction between a neutron and a proton cannot be different from that in any coherent superposition of both. Under these assumptions for isospin invariance, the functions $V_\alpha$ in (3.8) must be scalars in the

isospin space in the form

$$V_\alpha = V_{\alpha 0} + V_{\alpha 1}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2. \tag{3.11}$$

Sometimes it is convenient to describe the spin and isospin dependence of the NN interaction in terms of projection operators. We will show in the following that the terms of (3.8) can be derived in a more physically transparent way. For example, the spin part of the interaction can be written as

$$V_\sigma(r) \cdot \frac{1}{2}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \equiv V_\sigma(r) P_\sigma, \tag{3.12}$$

where $V_\sigma(r)$ describes the radial dependence and the operator $P_\sigma = \frac{1}{2}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$ has the expected values $+1$ for the triplet state and $-1$ for the singlet state. This can be shown starting from the vector $\mathbf{S} = \frac{\hbar}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$. Since $S^2 = \frac{\hbar^2}{4}(\sigma_1^2 + \sigma_2^2 + 2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$, then

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = \frac{1}{2}\left(-\sigma_1^2 - \sigma_2^2 + \frac{4S^2}{\hbar^2}\right); \tag{3.13}$$

the eigenvalues $\hbar^2 S(S+1)$ of $S^2$ are $+2\hbar^2$ for the triplet state ($S = 1$) and 0 for the singlet state. The eigenvalues of $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$ are equal to 3, so that the eigenvalues of $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ are equal to $+1$ for the triplet state and $-3$ for the singlet state, resulting in the expected values of $P_\sigma$ predicted above.

$P_\sigma$ is known as the *Bartlett potential* or *spin exchange potential* since, if we use for the spin functions similar to those given in table 3.1, we shall obtain that the operation of spin exchange is equivalent to multiplication by a factor $+1$ for the triplet state and a factor $-1$ for the singlet state.

If one assumes that the nuclear force depends on the parity of the wavefunction that describes the two particles, then a way of expressing that dependence is by means of a term of the form

$$V_r(r) P_r, \tag{3.14}$$

referred to as the *Majorana potential*, which contains the operator $P_r$ that exchanges the space coordinates of the two particles. The eigenvalues of $P_r$ are $+1$ and $-1$, if the wavefunction is even or odd, respectively.

The isospin dependence of the interaction can also be defined by the quantity

$$V_t(r) \cdot \frac{1}{2}(1 + \mathbf{t}_1 \cdot \mathbf{t}_2) \equiv V_t(r) P_t, \tag{3.15}$$

where the operator $P_t$ changes the isospin of the two particles. The antisymmetry of the total wavefunction implies that $P_t$ is not independent of $P_\sigma$ and $P_r$, embodied in the relation

$$P_t = -P_\sigma P_r, \tag{3.16}$$

which can be verified easily by the application of both sides to (3.1). The operator $P_\sigma P_r$ is known as the *Heisenberg operator*.

Gathering the terms presented up to now, we can write the expression that represents the central part of the nucleon-nucleon potential:

$$V_C(r) = V_W(r) + V_r(r) P_r + V_\sigma(r) P_\sigma + V_t(r) P_t, \tag{3.17}$$

where the portion $V_W$, dependent only on $r$, is usually referred to as the *Wigner potential*.