

Nuclear Physics
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Chapter 1

The NN interaction

1.1 General notions

There are a number of general features of the interaction between nucleons that can be easily inferred from the phenomenology of nuclei. We briefly recall them in what follows. Note that we avoid here the discussion of the deep differences between the interaction between two particles when they are in the vacuum or in the nuclear medium. The features listed here characterise, qualitatively, both the interaction in the vacuum and in the medium.

Of course, emphasis should be put on the basic idea that nuclei are self-bound systems and the NN interaction should be mainly attractive (at least in some configurations of the two nucleons like the one having the quantum numbers of the deuteron).

- *Short range.* The short-range nature of the NN interaction is well-known. It can be simply understood by a comparison between the scaling of energies and radii in atoms and nuclei. Atoms are governed by the long-range Coulomb force and for instance in hydrogen-like atoms the radius scales as $1/Z$, and the electron binding energy scales as Z^2 . If we look at the binding energy per particle BE/A in light nuclei we observe the behaviour of Table 1.1 (nuclei that undergo β -decay are excluded from this Table).

The α particle has already a value of BE/A that resembles the one at which there is saturation for most of the nuclei of the isotope chart. This can be interpreted by saying that nuclei saturate their BE/A because each nucleon interacts only with its nearest neighbours, and

Nucleus	BE/A [MeV]
${}^2\text{H} \equiv d$	1.112
${}^3\text{He}$	1.493
${}^4\text{He} \equiv \alpha$	7.073

Table 1.1: Binding energies per particle of the lightest nuclei

this happens already, roughly speaking, when four nucleons are bound together. If so, we can make a first estimate of the range of the NN force, which should be of the radius of the α particle. Looking at the experimental finding, we can say that

$$\text{Range} \approx 1.5 \text{ fm.} \quad (1.1)$$

- *Spin dependence.* The two nucleon system can be in the three different states pp, nn and pn. Only the pn system displays a bound state. How do we reconcile this fact with the charge independence property of the nuclear force, that is known to hold since the Coulomb force and charge-breaking terms of the NN force are relatively weak ? The solution lies in the spin dependence of the NN force, together with the role of the Pauli principle.

Formally the NN two-body potential commutes with isospin,

$$[V_{NN}, T^2] = 0, \quad [V_{NN}, T_z] = 0. \quad (1.2)$$

The consequence is that the two nucleon system must be in a state with good quantum numbers T, T_z . Since the neutron and proton states are, respectively, $|t = 1/2, t_z = 1/2\rangle$ and $|t = 1/2, t_z = -1/2\rangle$, the two-nucleon system can in principle be in the states $T = 0, 1$. However, for the pp and nn systems T_z is equal to 1 and -1 respectively, and thus these two systems can only be found in the $T = 1$ state. On the other hand, the pn system can also be in a $T = 0$ state. The other quantum numbers for the two-particle states can be the space angular momentum L and spin S . At low energy we can limit our discussion to $L = 0$ states. Since the total wave function must be antisymmetric, and the $L = 0$ part is symmetric, it is clear that $T = 1$ implies $S = 0$, while $T = 0$ implies $S = 1$. The NN interaction in the $T = 1, S = 0$ channel must be equal in the pp, nn, and pn cases, and it turns out

to be not strong enough to bind the two particles. The pn system can also exist in the $T = 0, S = 1$ state and has a bound state. Therefore we can deduce that

$$V(S = 1, T = 0) \text{ is more attractive than } V(S = 0, T = 1). \quad (1.3)$$

- *Spin-orbit component.* This component must exist in the NN interaction since we need it in many contexts. The magic numbers of nuclei cannot be explained without a strong spin-orbit force (having opposite sign with respect to the atomic case).
- *Tensor component.* The deuteron is not spherical but has an electric quadrupole moment. We also need a component of the NN interaction that explains this phenomenon. As will be discussed in the following, only the so-called tensor component can be advocated.

1.2 Structure of the NN interaction

In a phenomenological approach, we can postulate that the two-body NN interaction is the most general function of the position, momentum and spin of the two particles that fulfils the requirements of being invariant under translations, Galilean boosts, rotations, parity and time-reversal, as well as under the exchange of particles 1 and 2. The consequences of these requirements are best seen in the center-of-mass system in which the coordinates read

$$\begin{cases} \mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \\ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \end{cases} \quad (1.4)$$

with the corresponding conjugate momenta

$$\begin{cases} \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \\ \mathbf{p} = \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}. \end{cases} \quad (1.5)$$

The potential is therefore in principle a function

$$V(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, \sigma_i), \quad (1.6)$$

with $i = 1, 2$. Translations (Galilean boosts) are transformations in which $\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{a}$ ($\mathbf{p}_i \rightarrow \mathbf{p}_i + \mathbf{p}_0$). They do not change $\mathbf{r}$ and $\mathbf{p}$; if the potential must be invariant under them, it must be independent both of $\mathbf{R}$, that becomes $\mathbf{R} + \mathbf{a}$, and of $\mathbf{P}$, that becomes $\mathbf{P} + 2\mathbf{p}_0$. Rotational invariance implies that the potential must be a scalar. With two of the vectors $\mathbf{A}$, $\mathbf{B}$ among those that we have at our disposal $(\mathbf{r}, \mathbf{p}, \sigma_i)$, we can form

$$\begin{cases} \mathbf{A} \cdot \mathbf{B}, & \text{scalar product} \\ \mathbf{A} \pm \mathbf{B}, \mathbf{A} \times \mathbf{B} & \text{sum or difference and vector product} \\ \mathbf{A} \otimes \mathbf{B} \equiv \frac{1}{2}(A_i B_j + A_j B_i) - \frac{1}{3}\delta_{ij} \mathbf{A} \cdot \mathbf{B}. & \text{rank two-tensor product or tensor product} \end{cases} \quad (1.7)$$

The first two products are well-known from elementary courses, while the third one is defined and will be called simply tensor product in what follows. Since we have the operators $\mathbf{r}$ and $\mathbf{p}$ acting on space coordinates, and the spin operators acting on spin coordinates, we can say that the simplest way to form a scalar V is to combine a scalar, vector, or tensor formed with the space operators, with a scalar, vector, or tensor, respectively, formed with spin operators. The combination must respect parity and time-reversal invariance that we have not used yet.

We remind that under parity P ,

$$\begin{cases} \mathbf{r} \rightarrow -\mathbf{r} \\ \mathbf{p} \rightarrow -\mathbf{p} \\ \sigma_i \rightarrow \sigma_i, \end{cases} \quad (1.8)$$

whereas under time-reversal

$$\begin{cases} \mathbf{r} \rightarrow \mathbf{r} \\ \mathbf{p} \rightarrow -\mathbf{p} \\ \sigma_i \rightarrow -\sigma_i. \end{cases} \quad (1.9)$$

We label by $+$ ($-$) combinations of operators that are even (odd) under a given transformations.

From Table 1.2 we now construct the simplest possible combinations.

- Scalar-scalar combinations.

1. A simple function of r^2 (or r) is possible. We call it *central potential* $V_C(r)$. In principle it can be also a function of p .

	P	T		P	T
r^2	+	+	1	+	+
p^2	+	+	$\sigma_1 \cdot \sigma_2$	+	+
$\mathbf{r} \cdot \mathbf{p}$	+	-			
$\mathbf{r}$	-	+	$\sigma_1 \times \sigma_2$	+	+
$\mathbf{p}$	-	-	$\sigma_1 \pm \sigma_2$	+	-
$\mathbf{r} \times \mathbf{p}$	+	-			
$\mathbf{r} \otimes \mathbf{r}$	+	+	$\sigma_1 \otimes \sigma_2$	+	+
$\mathbf{p} \otimes \mathbf{p}$	+	+			
$\mathbf{r} \otimes \mathbf{p}$	+	-			

Table 1.2: Scalar, vector and tensor combinations of the space and spin operators, with their properties under transformations P and T .

2. A function V_S of r (or, in principle, of p) can be multiplied by $\sigma_1 \cdot \sigma_2$. We call it *spin potential*: $V_S \sigma_1 \sigma_2$.
- Vector-vector combinations.
3. The only possibility is to combine $\mathbf{r} \times \mathbf{p}$ with $\sigma_1 + \sigma_2$ (note that combining with $\sigma_1 - \sigma_2$ is not possible because it would not provide a term which is invariant under exchange $1 \leftrightarrow 2$). A function of scalars V_{LS} multiplied by $2(\mathbf{r} \times \mathbf{p})(\sigma_1 + \sigma_2) \equiv \mathbf{L} \cdot \mathbf{S}$ is called *spin-orbit potential*.
- Tensor-tensor combinations.
4. We can consider

$$\begin{aligned}
[\mathbf{r} \otimes \mathbf{r}][\sigma_1 \otimes \sigma_2] &= \left(\frac{1}{2} r_i r_j + \frac{1}{2} r_j r_i - \frac{1}{3} \delta_{ij} r^2 \right) \left(\frac{1}{2} \sigma_{1i} \sigma_{2j} + \frac{1}{2} \sigma_{1j} \sigma_{2i} - \frac{1}{3} \delta_{ij} \sigma_1 \cdot \sigma_2 \right) \\
&= \frac{1}{4} (\mathbf{r} \cdot \sigma_1) (\mathbf{r} \cdot \sigma_2) \times 4 - \frac{1}{3} \frac{1}{2} r^2 \sigma_1 \cdot \sigma_2 \times 2 \times 2 + \frac{1}{9} 3 r^2 \sigma_1 \cdot \sigma_2 \\
&= (\sigma_1 \cdot \mathbf{r}) (\sigma_2 \cdot \mathbf{r}) - \frac{1}{3} \sigma_1 \cdot \sigma_2 r^2 \equiv \frac{r^2}{3} S_{12}(\hat{r}), \tag{1.10}
\end{aligned}$$

where in the last step the so-called tensor operator S_{12} has been defined. A function of other scalars V_T times this quantity $S_{12}(\hat{r})$ is called *tensor potential*. In principle we can form the same combination with the momentum p , and we can introduce another function $V_{T'}$ times $S_{12}(\hat{p})$.

We can then ask ourselves the question whether more complicated combinations than the product of a scalar, vector or tensor in space with the same quantity in spin can appear. In principle we should enlarge our investigation. For instance, we may start by asking the question whether powers of scalars can produce new terms in the NN potential. It is clear that powers of r^2 , p^2 do not provide new structures. The same can be said for powers of $\sigma_1 \cdot \sigma_2$ (try the exercise yourself by reminding that $\sigma^2 = 3$ because the square of each Pauli matrix is 1). On the other hand, we could try to introduce two new combinations.

5. $(\mathbf{r} \cdot \mathbf{p})^2$ generates a possible new term of the potential because it is invariant under P and T . From simple vector algebra,

$$\begin{aligned} L^2 &= (\mathbf{r} \times \mathbf{p}) (\mathbf{r} \times \mathbf{p}) = r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2 \Rightarrow \\ (\mathbf{r} \cdot \mathbf{p})^2 &= r^2 p^2 - L^2, \end{aligned} \quad (1.11)$$

so that a function of $(\mathbf{r} \cdot \mathbf{p})^2$ can be replaced by a function of L^2 . Consequently, we can introduce a function of other scalars V_L times L^2 .

6. By the same argument and from Table 1.2, we can introduce a function of other scalars times $L^2 \sigma_1 \cdot \sigma_2$.

It is rather tedious to check if new combinations appear or not by combining more vectors or more tensors. We state without proof that the only new term that appears is

7. a so-called *quadratic spin-orbit potential* given by a function V_Q of other scalars times the combination $(\mathbf{L} \cdot \mathbf{S})^2$. This combination arises from $(\mathbf{r} \cdot \mathbf{p})(\mathbf{r} \otimes \mathbf{p})(\sigma_1 \otimes \sigma_2)$.

As a conclusion, the general form of the NN potential $V_{NN}(1, 2)$ turns out to be the sum of all the terms we have described. For each term we have a pp (or nn) plus a pn term. If we work in isospin representation, for each term we have the possibility to multiply by one or by $\tau_1 \cdot \tau_2$. This gives 14 terms. One of the most famous and widely used potentials, the so-called Argonne potential AV18 contains 18 terms because there is some charge-symmetry breaking term. This potential has been fitted using 1787 pp data and 2154 np data, related to scattering at laboratory energy smaller than 350 MeV. The reduced χ^2 is close to 1, testifying to the great accuracy of such potential.

1.3 The microscopic NN interaction

1.3.1 A pedagogical warm-up: scalar mesons and the Yukawa potential

In a microscopic picture, the interaction between particles arises from the exchange of another particle that acts as a mediator, that is, of a particle that transfers the energy and the quantum numbers associated with the interaction. Field theory is the modern theory of all forces when they are described along this line, with fields associated with the interacting particles and with the mediators. In the best known field theory, quantum electrodynamics (QED), photons are the bosons that mediate the electromagnetic force between elementary charged particles like electrons. The strong force is a more challenging case, since nucleons (like other particles that feel the strong force, i.e. hadrons) are not elementary. We do not aim in these lectures to describe the strong force at an elementary level, by considering the fact that nucleons are composed by quarks and gluons. This is the subject of quantum chromodynamics (QCD) and this theory is unfortunately highly non-perturbative at the low-energy scale which is of interest for nuclear structure and reactions. Thus, we rather consider nucleons as point-like and the interactions as mediated by mesons. This is less fundamental but more pedagogical and more tightly bound to the properties of the NN interactions that we have discussed above.

Before we attack the real description of the meson exchange between nucleons, we discuss a simplified problem in which the quantum numbers of the mediator are neglected. In other words, we discuss the interaction between nucleons by assuming that the simplest possible meson, a fictitious scalar-isoscalar one, is exchanged. In this way we accomplish two goals: (i) we discuss the basic tools of field theory and of the lowest order processes without further complications, and (ii) we see clearly the origin of the well-known Yukawa potential.

Strictly speaking, we could also introduce the Yukawa potential without this detour into the basics of field theory. Therefore, the reader can feel free to look at the next subsection, or to jump to the following one. Therein, the Yukawa potential is more simply deduced from the solution of the Klein-Gordon equation.

The Yukawa potential from second-order perturbation theory

Let us call ϕ the field associated with the scalar meson. The potential that couples the meson to the nucleons is assumed to be proportional to the field itself, namely

$$V = g\phi, \quad (1.12)$$

where g is a coupling constant. The probability amplitude for the various processes arising from such coupling can be represented through Feynman diagrams. By means of these diagrams we represent amplitudes that are associated with the various orders of perturbation theory. Time goes from the bottom to the top (in our conventions). The initial and final states are real states. The intermediate states are virtual ones, and can violate energy conservation by an amount ΔE (as they exist for times that can be shorter than $\approx \hbar/\Delta E$). In the diagrams of Fig. 1.1 the arrowed (wavy) lines represent a nucleon (meson), and at each vertex (defined as the point in which the meson line is emitted or absorbed by the nucleon line) the potential V is meant to act.

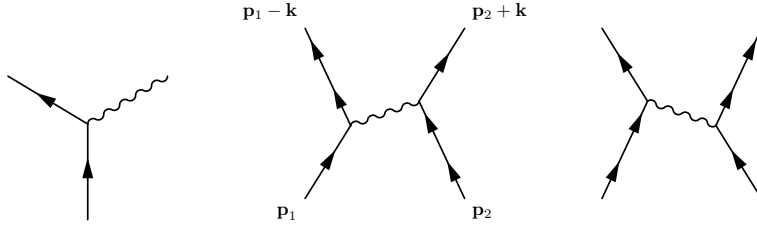


Figure 1.1: Feynman diagrams involving nucleons (lines with arrows) and mesons (wavy lines). See the text for a discussion.

The diagram on the left in Fig. 1.1 represents a process that, albeit first-order, cannot take place in low-energy nuclear physics since the meson has finite mass and consequently can only be created at the expense of energies which are larger than 100 MeV or even 700-800 MeV¹. The central and right diagrams in Fig. 1.1 correspond instead to processes that can take place and are second-order ones. We will denote by n, m, i the final, intermediate and initial state, respectively, in those diagrams. The second order expression for

¹These are the masses of the real mesons that can be exchanged, cf. the discussion that is carried out below.

the energy in perturbation theory reads

$$\Delta E = \sum_m \frac{\langle n|V|m\rangle\langle m|V|i\rangle}{E_i - E_m}. \quad (1.13)$$

We calculate this expression by assuming that ingoing and outgoing nucleons are free particles and can be represented by plane waves. We label the momenta as in the figure. The meson will be labelled by a given momentum as well, that is, in the central diagram of Fig. 1.1 the momentum $\mathbf{k}$ is exchanged and the meson field will be $\phi_{\mathbf{k}}$. Note that in the diagram at left momentum conservation implies that we must deal with $\phi_{-\mathbf{k}}$ ².

The meson field can be seen as a generalised coordinate, namely $\phi_{\mathbf{k}}$ represents the amplitude for a meson eigenmode having momentum $\mathbf{k}$. As in the standard harmonic oscillator, in which the amplitude is written

$$x = \sqrt{\frac{1}{2m\omega}} (a + a^\dagger),$$

we also can safely say that $\phi_{\mathbf{k}}$ will be a superposition of a creation operator of a mode $\mathbf{k}$ and an annihilation operator of a mode $-\mathbf{k}$. The coefficient in front of such a superposition can be deduced by means of the following procedure: (i) imagine the free field obey the equation for free mesons, that is, the Klein-Gordon equation; (ii) write the Lagrangian or Hamiltonian from which such an equation is deduced (as Euler-Lagrange equation); (iii) transform this Hamiltonian as a function of ϕ and of the conjugate momentum into a function of creation and annihilation operators; (iv) choose the coefficients in such a way that the Hamiltonian becomes $H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$. This procedure can be found in any standard text of quantum field theory and will not be repeated here. The result is that

$$\phi_{\mathbf{k}} = \frac{1}{\sqrt{\Omega}} \frac{1}{\sqrt{\omega_{\mathbf{k}}}} (a_{-\mathbf{k}} + a_{\mathbf{k}}^\dagger), \quad (1.14)$$

where Ω is the quantisation volume and we remind that the energy is given here as $\omega_{\mathbf{k}} = \sqrt{k^2 + \mu^2}$, μ being the meson mass.

In this way, the diagram in the centre of Fig. 1.1 has matrix elements given by

$$V_{mi} = \frac{g}{\sqrt{\Omega}} \frac{1}{\sqrt{\omega_{\mathbf{k}}}} \langle \mathbf{k} | a_{-\mathbf{k}} + a_{\mathbf{k}}^\dagger | 0 \rangle. \quad (1.15)$$

²We use natural units in this subsection, $\hbar = c = 1$.

This value is the same for V_{nm} as it can be easily verified. Then, the values of the two centre and right diagrams of Fig. 1.1 can be written as

$$\text{centre} = \frac{g^2}{\Omega} \frac{1}{\omega_{\mathbf{k}} \frac{\mathbf{p}_1^2}{2M} - \omega_{\mathbf{k}} - \frac{(\mathbf{p}_1 - \mathbf{k})^2}{2M}}, \quad (1.16)$$

$$\text{right} = \frac{g^2}{\Omega} \frac{1}{\omega_{\mathbf{k}} \frac{\mathbf{p}_2^2}{2M} - \omega_{\mathbf{k}} - \frac{(\mathbf{p}_2 + \mathbf{k})^2}{2M}}. \quad (1.17)$$

Since we want to study low-energy processes, we neglect the terms of the type $\mathbf{p}^2/2M$ or $(\mathbf{p} \pm \mathbf{k})^2/2M$ with respect to $\omega_{\mathbf{k}}^3$. Then, the sum of the two diagrams is

$$V(\mathbf{k}) = -\frac{g^2}{\Omega} \frac{2}{\omega_{\mathbf{k}}^2}. \quad (1.18)$$

This quantity is called $V(\mathbf{k})$ since it is actually the interaction energy between nucleons when a meson with momentum $\mathbf{k}$ is exchanged. But we still miss the sum over all possible intermediate states [the sum over m in Eq. (1.13)]: this is easy to perform since the density of states for the meson having momentum $\mathbf{k}$ in the quantisation volume is $\frac{\Omega}{(2\pi)^3}$. Thus,

$$V(\mathbf{k}) = -\frac{g^2}{4\pi^3 \omega_{\mathbf{k}}^2}. \quad (1.19)$$

The interaction potential is just the Fourier transform of this quantity, namely

$$V(\mathbf{r}) = -\frac{g^2}{4\pi^3} \int d^3k \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2 + \mu^2}. \quad (1.20)$$

This integral can be calculated using the residue theorem, and the result reads

$$V(\mathbf{r}) = -\frac{g^2}{2\pi} \frac{e^{-\mu r}}{r}. \quad (1.21)$$

This is the celebrated Yukawa potential. Therefore, we have learned that this potential arises if a scalar meson is exchanged and, by inspection, we see that this is a screened electrostatic potential that becomes like $\sim 1/r$ if the meson mass is zero.

³The kinetic energies of nucleons in nuclei, as it will be discussed below, range between 0 and ≈ 40 MeV, whereas the meson energies include the mass of those mesons that are either 135 MeV or 700-800 MeV.

The Yukawa potential from the Klein-Gordon equation

The Yukawa potential, derived within the approximations that have been discussed above, can also be simply deduced as the Fourier transform of the meson propagator that is associated with the solution of the Klein-Gordon equation. The Klein-Gordon equation that the meson wave function must satisfy reads

$$(\square + \mu^2) \phi = 0. \quad (1.22)$$

If we do not consider the time-dependence, the same equation becomes

$$(-\nabla^2 + \mu^2) \phi = 0, \quad (1.23)$$

and the associated equation for the propagator G reads

$$(\nabla^2 - \mu^2) G(\vec{r}, \vec{r}') = -\delta^{(3)}(\vec{r} - \vec{r}'). \quad (1.24)$$

The solution of this equation is the free meson propagator,

$$G(\vec{r}, \vec{r}') = \frac{e^{-\mu|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}. \quad (1.25)$$

This has the structure of the Yukawa potential.

1.3.2 The one-pion exchange potential

By looking at the meson table, we realise that the lightest meson that can be exchanged between nucleons is the pion. As compared with the fictitious scalar meson that we have discussed in 1.3.1, the pion (i) is pseudo-scalar and (ii) belongs to an isospin (or charge) triplet since it exists in the three states, $\pi^0, \pi^\pm$. Thus, (i) the pion vertex must include a scalar operator that changes parity: this is given by $\sigma \cdot \mathbf{k}$, since the momentum $\mathbf{k}$ is equal to $-i\nabla$ and the gradient changes the parity of any wave function; (ii) the pion exchange must include isospin operators. Therefore, we generalise Eq. (1.19) to

$$V(\mathbf{k}) = -\frac{g^2}{4\pi^3\omega_{\mathbf{k}}^2} (\sigma_1 \cdot \mathbf{k}) (\sigma_2 \cdot \mathbf{k}) (\tau_1 \cdot \tau_2). \quad (1.26)$$

Conventionally, one writes

$$\begin{aligned} V(\mathbf{r}) &= -\frac{g^2}{2\pi} (\sigma_1 \cdot \mathbf{k}) (\sigma_2 \cdot \mathbf{k}) \frac{e^{-\mu r}}{r} (\tau_1 \cdot \tau_2) \\ &= \frac{f_\pi^2}{m_\pi^2} (\sigma_1 \cdot \nabla) (\sigma_2 \cdot \nabla) \frac{e^{-m_\pi r}}{r} (\tau_1 \cdot \tau_2). \end{aligned} \quad (1.27)$$

This quantity can be calculated by using standard vector algebra, in particular $\nabla_j(1/r) = -(r_j/r^3)$ and $\nabla_i(-r_j/r^3) = 3r_i r_j/r^5$ if $i \neq j$, together with $\nabla_i \nabla_j(1/r) = -(4\pi/3)\delta(r)$ if $i = j$. The result reads

$$V_{\text{OPEP}}(r) = f_\pi^2 m_\pi \tau_1 \cdot \tau_2 \left[S_{12} \left(\frac{1}{(m_\pi r)^3} + \frac{1}{(m_\pi r)^2} + \frac{1}{(3m_\pi r)} \right) e^{-m_\pi r} + \frac{\sigma_1 \cdot \sigma_2}{3} \frac{e^{-m_\pi r}}{m_\pi r} - \frac{4\pi}{3m_\pi^3} \delta(r) \sigma_1 \cdot \sigma_2 \right]. \quad (1.28)$$

The last term [i.e. the contact one, proportional to $\delta(r)$] can be dropped because we will show below that other terms dominate at short distances. The remaining terms form the so called one-pion exchange potential (OPEP). This potential has only spin and tensor terms and its range is $1/m_\pi$ that becomes, resorting to usual units, $\hbar/m_\pi c = 1.4$ fm.

The spin part is attractive (repulsive) in the $S = 0$ ($S = 1$) channel since it is known that

$$\langle S | \sigma_1 \cdot \sigma_2 | S \rangle = \begin{cases} -3 & S = 0; \\ +1 & S = 1. \end{cases} \quad (1.29)$$

The tensor part depends on the expectation value of

$$S_{12} = \begin{cases} -1 & \text{if the spins are orthogonal to } \hat{r} \text{ (oblate configuration);} \\ +2 & \text{if the spin are parallel to } \hat{r} \text{ (prolate configuration).} \end{cases} \quad (1.30)$$

The meaning of oblate or prolate configurations may become more transparent by inspection of Fig. 1.2.

We now make a rough estimate of the order of magnitude of the potential (1.28), by taking into account the fact that experimentally one finds⁴ that $f_\pi^2 = 0.081$. The estimate is made at values of r of the order of m_π and considering only the spin term, proportional to $\sigma_1 \sigma_2$, in the case $S = 1$ and $T = 0$ [cf. Eq. (1.29) and an analogous one for T]:

$$V_{\text{OPEP}} \approx 0.081 \times 135 \times (-3) \frac{1}{3e} \approx -5 \text{ MeV}. \quad (1.31)$$

The kinetic energy associated with the relative motion of two nucleons at the same distance can be, in turn, estimated from the uncertainty principle⁵:

$$\Delta p \approx \frac{\hbar}{\Delta r} \Rightarrow \frac{(\Delta p)^2}{2M} = \frac{(\hbar c)^2}{2Mc^2 (\Delta r)^2} \approx 10 \text{ MeV}. \quad (1.32)$$

⁴From pion-nucleon experiments.

⁵We go back to usual units from here on.

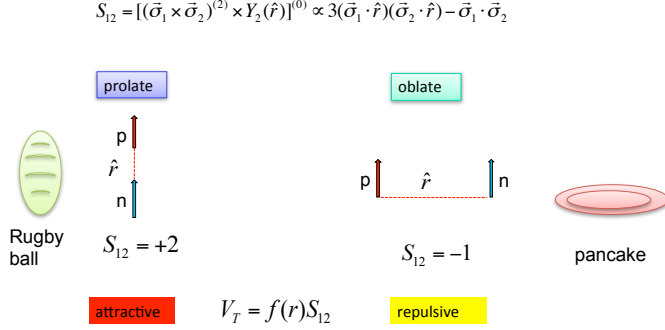


Figure 1.2: Schematic picture of the expectation values of the tensor operator S_{12} when the spins are either aligned with (prolate configuration) or perpendicular to (oblate configuration) the relative distance vector $\mathbf{r}$. The function $f(r)$ is negative, favouring a prolate shape for the deuteron, as it is clear from Eq. (1.28) when one takes care of the fact that the deuteron has isospin $T = 0$ in its ground state.

As the attractive part of the spin-spin potential is smaller than the kinetic energy, it cannot bind the two-nucleon system. The tensor part is definitely more attractive, as we can estimate it to be

$$V_{\text{OPEP}} \approx 0.081 \times 135 \times (-3)(+2) \frac{7}{3e} \approx -50 \text{ MeV} \quad (1.33)$$

in the case $T = 0$ and in the prolate configuration. In the oblate case it would be instead repulsive, and at the moment we cannot decide which is the relative weight of oblate and prolate configurations.

1.3.3 Two pion exchange potential

The main shortcomings of the OPEP are: (i) it lacks a central part, and (ii) it is unclear whether it provides enough binding. In more general terms than just for the deuteron, the OPEP vanishes when averaged over different

spin configurations (the average over isospin will vanish as well). We ask ourselves the question whether a two-pion exchange potential can avoid such limitations.

One indeed expects a two-pion exchange potential with range $\sim 1/(2m_\pi) \approx 0.7$ fm. Its microscopic calculation is very tough since a huge number of diagrams could be involved. Some of them are drawn in Fig. 1.3. We do not attempt detailed estimates. Rather, we represent the amplitudes associated with the diagrams in Fig. 1.3 as “squares” of the one-pion exchange potential divided by the energy denominator associated with the difference between initial and intermediate state, according to Eq. (1.13). In some of these diagrams the intermediate state is close in energy to the initial state, while if one Δ -isobar is excited, the energy is increased by 300 MeV. We take an average energy denominator $\bar{E} \approx 150$ MeV.

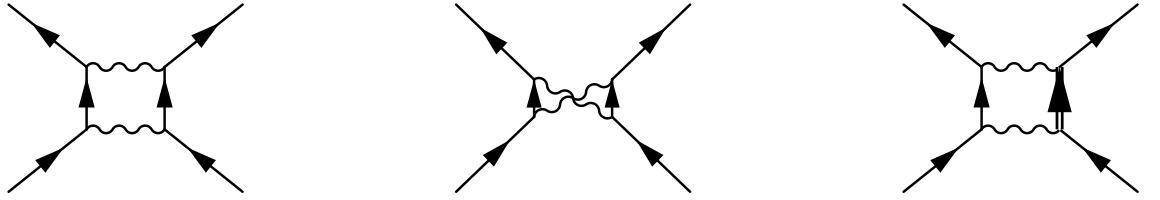


Figure 1.3: Few examples of Feynman diagrams involving two pions exchange. The double arrowed line denotes a nucleon internal excitation like, for instance, a Δ -isobar.

We do not discuss in detail the potential

$$V_{2\pi} = -\frac{V_{\text{OPEP}}^2}{\bar{E}}. \quad (1.34)$$

However, we use the equations (easy to deduce from operator algebra) showing that

$$(\tau_1 \cdot \tau_2)^2 = 3 - 2\tau_1 \cdot \tau_2,$$

and

$$S_{12}^2 = 6 + 2\sigma_1 \cdot \sigma_2 - 2S_{12},$$

to support the statement that the potential (1.34) contains, in addition to spin and tensor terms that are prone to average to zero, central terms as well (it also contains isoscalar terms in addition to isovector terms).

We shall then make an estimate of the importance of the two-pion exchange vs. the one-pion exchange, along the same line as in Eq. (1.33). We first identify the structure of the two-pion potential by writing

$$V_{2\pi} = -\frac{1}{E} f_{\pi}^4 m_{\pi}^2 (3 - 2\tau_1 \cdot \tau_2) \left[(6 + 2\sigma_1 \cdot \sigma_2 - 2S_{12}) f^2(r) + \frac{1}{9} (3 - 2\sigma_1 \cdot \sigma_2) g^2(r) + 2S_{12} \frac{\sigma_1 \cdot \sigma_2}{3} f(r)g(r) \right], \quad (1.35)$$

where f and g identify the radial functions associated with the first two terms in the square brackets of Eq. (1.28) (we remind that the zero-range term, namely the last one, has been dropped). By inspection, we immediately see that the two-pion potential has a central component.

If we drop the terms including S_{12} to avoid estimating the weight of oblate and prolate components,

$$V_{2\pi} = -\frac{1}{E} f_{\pi}^4 m_{\pi}^2 (3 - 2\tau_1 \cdot \tau_2) \left[(6 + 2\sigma_1 \cdot \sigma_2) f^2(r) + \frac{1}{9} (3 - 2\sigma_1 \cdot \sigma_2) g^2(r) \right], \quad (1.36)$$

and the value of the first term in square brackets of (1.36), estimated in the $S = 1, T = 0$ state at a distance corresponding to the OPEP range [so that we can compare with (1.33)], reads

$$V_{2\pi} \approx -\frac{1}{150} 0.081^2 \times 135^2 \times 9 \left(8 \times \frac{49}{9e^2} \right) \approx -40 \text{ MeV}. \quad (1.37)$$

The second term in square brackets provides a smaller contribution. This force is large enough to bind the two-nucleon system, namely the deuteron.

1.3.4 The short-range repulsion

We treat this part in a less mathematical, more intuitive fashion. The pion is the lightest meson that can be exchanged among nucleons. Another possible exchange among nucleons is that of the ρ and ω mesons: their mass is around 700-800 MeV, so that the corresponding range should be around 0.3 fm or less. These mesons have angular momentum and parity 1^- so they are vector mesons. The structure of the interaction they mediate is expected to be similar to the structure of the electromagnetic force since the photon has the same quantum numbers. Consequently, we draw conclusions without resorting to any Feynman diagram calculation:

- the main part of the interaction will be like the Coulomb force between two identical charges, namely a strongly repulsive one;
- we expect the emergence of a spin-orbit force of the type

$$\frac{1}{r} \frac{dV}{dr} \mathbf{L} \cdot \mathbf{S}. \quad (1.38)$$

Since V is repulsive, the spin-orbit force has the opposite sign than in the atomic case, and because of the large absolute value of V and of its derivative, it is also a much more intense force: as discussed in the next Chapter, it creates the difference between the atomic and nuclear shell closures.

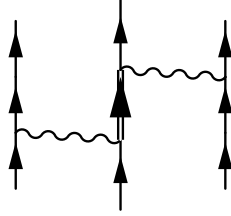


Figure 1.4: Feynman diagram corresponding to one of the most important contributions to the three-body force, that is, the so-called Fujita-Miyazawa process. The intermediate double line represents a Δ -isobar.

1.3.5 The three-body force

Three body forces appear in nuclei as in many other quantum systems. They reflect the existence of elementary processes that involve more than one pair of particles. In the nuclear case, the so-called Fujita-Miyazawa process depicted in Fig. 1.4 is believed to be the most important process. It is analogous to the force with which tides on the earth surface affect the motion of a satellite, since tides are caused by the effect of the moon. In other words, we can imagine that the three lines of Fig. 1.4 are, from left to right respectively, the moon, the earth and the satellite: the first interaction line represent the fact

that the moon induces tides, if we allow ourselves to imagine the Δ -isobar excitation of the nucleon as the earth with tides, and the second interaction line is the effect of these induced tides on the satellite.

As soon as one moves from the two-body nucleon system to the three body systems, whether bound systems like ^3H or ^3He or reactions like (d,p) etc., it turns out that no sensible theory can be made without three-body forces: any model including only two-body forces will not fit the experimental data. The accurate NN potentials mentioned at the end of Sec. 1.2 must be complemented with three-body forces.

The improvement of three-body forces and the quest for possible four-body forces are active fields of research in nuclear physics.

Chapter 2

The nuclear shell structure: empirical potentials and microscopic theories

2.1 Generalities on the evidence of a nuclear mean-field

Early experiments have already proven that in a nucleus the mean free path of a nucleon is quite large, namely of the order of the nuclear radius. For quite some time it has appeared difficult to accommodate this evidence into the picture of a strongly interacting system. In fact, the solution of the puzzle lies in the fact that the nucleus is not a very dense system. The nuclear radius is

$$R = r_0 A^{1/3}, \quad \text{with } r_0 = 1.2 - 1.3 \text{ fm.} \quad (2.1)$$

Consequently, the volume per nucleon is

$$\frac{V}{A} = \frac{\frac{4}{3}\pi r_0^3 A}{A} \approx 4r_0^3 \quad (2.2)$$

and the average distance between nucleons is of the order of ≈ 2 fm. In such conditions, nucleons do not feel during most of their time the strong repulsive part of the NN interaction that has much shorter range as we know from the previous Chapter. Not only nucleons remain at a relatively large distance from one another; on top of this, experiments show that the density

is approximately constant within the nuclear volume, and so we can expect that each of them feels a relatively smooth potential which is an average of the interactions with all the others. Only at the edge of the nucleus, where the density drops, the attractive potential drops as well. So, the profiles of the potential and of the density follow each other as shown schematically in Fig. 2.1.

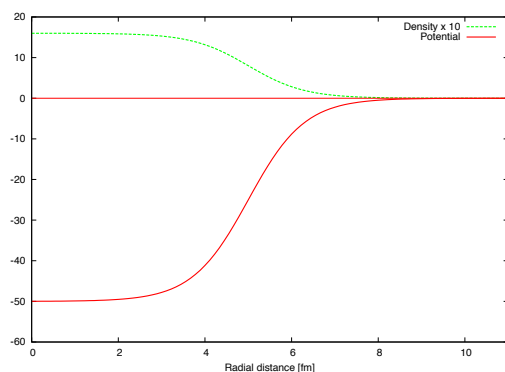


Figure 2.1: Schematic profiles of density (positive y-axis) and potential (negative y-axis) in a nucleus. The density has dimensions fm^{-3} but it has been multiplied by a factor 10 in the figure, while the potential has dimensions of MeV.

All the nuclear phenomenology, starting from the evidence from the binding energies, favours the above picture in the sense that we know that nucleons occupy orbitals that are grouped in shells, like in the atomic case. The nuclear shells differ from the atomic shells as we discuss below. The main point to be kept in mind is that the atomic shells are mainly governed by an external potential (the Coulomb potential of the atomic nucleus), while nu-

2.1. GENERALITIES ON THE EVIDENCE OF A NUCLEAR MEAN-FIELD23

clear shells are emerging from the complicated behaviour of self-interacting particles. An empirical potential that reproduces well the nuclear phenomena is the Woods-Saxon potential. In the case of spherical nuclei, this reads

$$U(r) = \frac{U_0}{1 + e^{\frac{r-R}{a}}}, \quad (2.3)$$

where R is the nuclear radius given by Eq. (2.1), a is a parameter that defines the nuclear surface and is called diffuseness (its value lies around 0.5-0.6 fm), and U_0 is the potential depth that is typically around -50 MeV. This follows, as we previously said, the density profile that can be extracted from electron scattering and is consistent with a form like

$$\rho(r) = \frac{\rho_0}{1 + e^{\frac{r-R}{a}}} \quad (2.4)$$

where ρ_0 is $\approx 0.16 \text{ fm}^{-3}$.

In order to understand the shell structure created by such a potential we should solve numerically the Schrödinger equation associated with it. Since we have assumed the potential to be spherical, we know that the wave functions have the form

$$\psi(\mathbf{r}) = \frac{u(r)}{r} \left[Y_\ell(\hat{r}) \otimes \chi_{\frac{1}{2}}(\sigma) \right]_{jm}. \quad (2.5)$$

The radial wave function is the solution of the corresponding reduced Schrödinger equation,

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + U(r) \right] u = \varepsilon u, \quad (2.6)$$

and has eigenvalue ε . It is more pedagogical to approximate the Woods-Saxon potential with a harmonic oscillator one, $U = \frac{1}{2}m\omega r^2$, since the harmonic oscillator solutions are known analytically. We do not actually need to know the detailed expression of such wave functions (that can be found in a standard quantum mechanics textbook) but just the eigenvalues, that are

$$\varepsilon_{n\ell} = \hbar\omega \left(2n + \ell - \frac{1}{2} \right) \equiv \hbar\omega \left(N + \frac{3}{2} \right), \quad (2.7)$$

where n assumes the values $n = 1, 2, 3 \dots$ and $n-1$ corresponds to the number of nodes of the radial wave function, $\ell = 0, 1, 2 \dots$, and $N = 2(n-1) + \ell$ is called the principal quantum number.

NOTE FOR STUDENTS: with the figure that each of you should have, or that you can see as Fig. 5.9 of the textbook (see web page) you can follow the discussion below. The harmonic oscillator levels, that appear at the left in the figure, possess a large degree of degeneracy. This is broken since the actual potential (2.3) is not the harmonic potential (due to this, for instance, the 1f levels are on average lower than the 2p levels). This breaking does not alter the shell structure. Quantitatively more important, instead, is the spin-orbit potential that shifts $j = \ell + 1/2$ levels downward and $j = \ell - 1/2$ levels upward. Once this level sequence is defined, the nucleons are disposed on the levels from the bottom up, by respecting the degeneracy of each level which is $2j + 1$ (it would be $2(2\ell + 1)$ for the levels with the same ℓ and different j when they are degenerate). This concept is very crucial since using it, by knowing the numbers of neutrons and protons, we immediately see if a system is closed shell or open shells. We now discuss the typical value of the shell gap $\hbar\omega$.

It may be useful to recall that the expectation value of the square of the nuclear radius in the harmonic oscillator potential is given by

$$\langle r^2 \rangle_{N\ell} = \frac{\hbar}{m\omega} \left(N + \frac{3}{2} \right) \quad (2.8)$$

It is possible to use this fact in order to decide what is the optimal value for the harmonic oscillator frequency ω . Let us consider a nucleus whose levels are filled up to the shell characterised by the principal quantum number $N_{\max}$. For even values of N , $\ell = 0, 2 \dots N$ while for odd values $\ell = 1, 3 \dots N$. In both cases the degeneracy of the shell is given by the sum over the allowed values of ℓ of the degeneracies $2(2\ell + 1)$. It can be shown that, irrespective of the parity of N , one can derive the total shell degeneracy as

$$\sum_{\ell} 2(2\ell + 1) = (N + 2)(N + 1). \quad (2.9)$$

The total number of particles can also be determined as a function of N by means of an elementary calculation and turns out to be

$$\sum_{N=0}^{N_{\max}} (N+1)(N+2) = \frac{1}{3}(N_{\max}+1)(N_{\max}+2)(N_{\max}+3) \approx \frac{1}{3}(N_{\max}+2)^3 \quad (2.10)$$

where the last approximation holds for large values of $N_{\max}$. The average square radius of such a nucleus can be obtained by making the weighted

average of the square radii of the different shells, where the weight is of course the number of particles in the shell. To simplify, we consider an equal number of protons and neutrons that contribute equally to the radius, and this brings an additional factor 2, namely

$$\begin{aligned}\langle r^2 \rangle &= \frac{2}{A} \sum_{N=0}^{N_{\max}} \frac{\hbar}{m\omega} \left(N + \frac{3}{2} \right) (N+1)(N+2) \\ &\approx \frac{1}{A} \frac{\hbar}{2m\omega} (N_{\max} + 2)^4.\end{aligned}\tag{2.11}$$

The sum could be performed exactly but the second line of the previous equation provides a reasonable approximation for large $N_{\max}$. If we now consider the easy relations

$$\langle r^2 \rangle = \frac{3}{5} R^2 = \frac{3}{5} r_0^2 A^{2/3},$$

then Eq. (2.11) can be solved to give the still unknown value of $\hbar\omega$:

$$\hbar\omega \approx \frac{5}{4} \left(\frac{3}{2} \right)^{1/3} \frac{\hbar^2}{mr_0^2} A^{-1/3} \approx 41 A^{-1/3} \text{ MeV}.\tag{2.12}$$

2.2 Hartree-Fock theory

The goal of the Hartree-Fock (HF) theory is to give a microscopic foundation, starting from the Hamiltonian formalism, to the existence of a nuclear mean field. Let us then start from an Hamiltonian

$$H = T + V,\tag{2.13}$$

where

$$T = \sum_{\alpha} -\frac{\hbar^2}{2m} \nabla_{\alpha}^2\tag{2.14}$$

is the kinetic energy, and

$$V = \frac{1}{2} \sum_{\alpha \neq \beta} v(\mathbf{r}_{\alpha}, \mathbf{r}_{\beta})\tag{2.15}$$

is a two-body interaction between nucleons. For the sake of simplicity, we omit here the spin dependence. The assumption of the HF theory is that

the total wave function is the totally anti-symmetric product of (in general unknown) single-particle wave functions. If these single-particle wave functions are labelled by ϕ_i , then the total wave function is given by the so-called Slater determinant

$$\begin{aligned}\Psi(\mathbf{r}_1 \dots \mathbf{r}_A) &= \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_1(\mathbf{r}_1) & \dots & \phi_A(\mathbf{r}_1) \\ \dots & \dots & \dots \\ \phi_1(\mathbf{r}_A) & \dots & \phi_A(\mathbf{r}_A) \end{vmatrix} \\ &= \frac{1}{\sqrt{A!}} \sum_P (-)^P \phi_{P(1)}(\mathbf{r}_1) \dots \phi_{P(A)}(\mathbf{r}_A),\end{aligned}\quad (2.16)$$

where in the last line we introduce the sum over all possible permutations P of the indices $1 \dots A$, and $(-)^P$ is $+1$ (-1) if the number of exchanges with respect to the identical permutation is even (odd). Consequently, the Hartree-Fock theory amounts to the minimisation of the total energy in the Hilbert space made up with such kind of wave functions, so that we need to calculate the expectation value $E = \langle \Psi | H | \Psi \rangle$ and to solve the equation

$$\delta E = 0. \quad (2.17)$$

We will actually add a Lagrange multiplier to ensure orthonormal wave function, so that the equations will be

$$\delta \left(E - \lambda \int d^3r \phi_i^*(\mathbf{r}) \phi_i(\mathbf{r}) \right). \quad (2.18)$$

We first calculate the expectation value of the kinetic energy and we obtain

$$\langle \Psi | T | \Psi \rangle = \sum_{i=1}^A -\frac{\hbar^2}{2m} \int d^3r \phi_i^*(\mathbf{r}) \nabla^2 \phi_i(\mathbf{r}). \quad (2.19)$$

The calculation of the expectation value of the potential can be developed along a similar line and one obtains

$$\begin{aligned}\langle \Psi | V | \Psi \rangle &= \frac{1}{2} \sum_{ij} \int d^3r \, d^3r' \, \phi_i^*(\mathbf{r}) \phi_j^*(\mathbf{r}') v(\mathbf{r}, \mathbf{r}') \phi_i(\mathbf{r}) \phi_j(\mathbf{r}') + \\ &- \frac{1}{2} \sum_{ij} \int d^3r \, d^3r' \, \phi_i^*(\mathbf{r}) \phi_j^*(\mathbf{r}') v(\mathbf{r}, \mathbf{r}') \phi_i(\mathbf{r}') \phi_j(\mathbf{r}).\end{aligned}\quad (2.20)$$

By making the variation defined by Eq. (2.18) using ϕ_i^* as a quantity to be varied (the variation of ϕ_i would not produce a new independent equation), one arrives at the following set of equations

$$\begin{aligned} & -\frac{\hbar^2}{2m} \nabla_i^2 \phi_i(\mathbf{r}) + \sum_{j=1}^A \int d^3r' \phi_j^*(\mathbf{r}') v(\mathbf{r}, \mathbf{r}') \phi_j(\mathbf{r}') \phi_i(\mathbf{r}) + \\ & - \sum_{j=1}^A \int d^3r' \phi_j^*(\mathbf{r}') v(\mathbf{r}, \mathbf{r}') \phi_j(\mathbf{r}) \phi_i(\mathbf{r}') = \varepsilon_i \phi_i(\mathbf{r}). \end{aligned} \quad (2.21)$$

One should note that the Lagrange multiplier λ introduced in Eq. (2.18) has been turned into ε_i . In fact, the set of equations (2.21) has the form of a set of Schrödinger equations and ε_i is the associated eigenvalue.

The difference with a set of ordinary Schrödinger equations is that the potential depends on the wave functions, and in this respect we call these a set of *self-consistent* equations. The solution must be found in an iterative fashion. One starts with a guess for some initial potential, finds the corresponding initial wave functions, insert them in the HF equations and iterate the procedure until convergence. At the l.h.s. of Eq. (2.21), after the term corresponding to the kinetic energy, one finds the Hartree and Fock terms. The Hartree term can be re-written as

$$\sum_{j=1}^A \int d^3r' \phi_j^*(\mathbf{r}') v(\mathbf{r}, \mathbf{r}') \phi_j(\mathbf{r}') \phi_i(\mathbf{r}) = \int d^3r' v(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') \times \phi_i(\mathbf{r}). \quad (2.22)$$

This term has a simple interpretation, since it represents the interaction of a nucleon with an environment where other nuclei are distributed with a density ρ . It also corresponds to a multiplicative potential and, despite it requires the self-consistent iterations, it does not affect the structure of the equation. The Fock term cannot be written as a multiplicative potential times the wave function, it rather renders the whole equation an integro-differential one and is due of course to the fermionic nature of the nucleons.

The binding energy is equivalent, within this model, to the total energy given by the sum of (2.19) and (2.20). We will use the notation

$$\langle i|t|j \rangle = -\frac{\hbar^2}{2m} \int d^3r \phi_i^*(\mathbf{r}) \nabla^2 \phi_j(\mathbf{r}) \quad (2.23)$$

for the matrix elements of the kinetic energy (and of any one-body operator), and

$$\langle ij|v|kl\rangle_{\text{AS}} = \int d^3r d^3r' \phi_i^*(\mathbf{r})\phi_j^*(\mathbf{r}')v(\mathbf{r},\mathbf{r}') [\phi_i(\mathbf{r})\phi_j(\mathbf{r}') - \phi_i(\mathbf{r}')\phi_j(\mathbf{r})] \quad (2.24)$$

for the antisymmetrized matrix elements of the two-body potential. Then, the total energy can be written

$$E = \sum_i \langle i|t|i\rangle + \frac{1}{2} \sum_{ij} \langle ij|v|ij\rangle_{\text{AS}}. \quad (2.25)$$

Sometimes one can use the notation $\langle ij|\bar{v}|kl\rangle$ instead of $\langle ij|v|kl\rangle_{\text{AS}}$ for the sake of further brevity.

2.2.1 Brief update on effective interactions for the nuclear Hartree-Fock

The HF equations cannot be solved easily with a potential having a strong short-range repulsion because in that case the Hartree term would diverge. In fact, in that case, the integral at the r.h.s. of Eq. (2.22) will in practice diverge for values of $r \approx r'$. Moreover, calculations based on any bare force point clearly to ground-state wave functions that are correlated, far from being like a Slater determinant. Therefore, one should build *ad hoc* interactions whose parameters are fitted to describe well the ground state of nuclei at the HF level. This has been extensively done in the last four decades. Popular effective interactions include the so-called Skyrme and Gogny forces, plus others. Skyrme-type interactions are zero-range forces while Gogny forces are of Gaussian type. These interactions reproduce, on average, the binding energies of nuclei with a remarkable accuracy of ≈ 1 -2 MeV.

We do not discuss the details here. It is only useful to recall that the popularity of Skyrme forces stems from the fact that with a force that is proportional to a Dirac delta function, $\delta(\mathbf{r} - \mathbf{r}')$, the Fock term loses its non-locality and becomes proportional to the Hartree term. The complete Skyrme interaction reads

$$\begin{aligned} v(\mathbf{r},\mathbf{r}') &= t_0(1+x_0P_\sigma)\delta(\mathbf{r}) + \frac{1}{2}t_1(1+x_1P_\sigma)[\mathbf{P}'^2\delta(\mathbf{r}) + \delta(\mathbf{r})\mathbf{P}^2] \\ &+ t_2(1+x_2P_\sigma)\mathbf{P}' \cdot \delta(\mathbf{r})\mathbf{P} + \frac{1}{6}t_3(1+x_3P_\sigma)\rho^\alpha(\mathbf{R})\delta(\mathbf{r}) \\ &+ iW_0(\sigma_1 + \sigma_2) \cdot [\mathbf{P}' \times \delta(\mathbf{r})\mathbf{P}] , \end{aligned} \quad (2.26)$$

where $\mathbf{r} = \mathbf{r} - \mathbf{r}'$, $\mathbf{R} = \frac{1}{2}(\mathbf{r} + \mathbf{r}')$, $\mathbf{P} = \frac{1}{2i}(\nabla - \nabla')$, $\mathbf{P}'$ is the hermitian conjugate of $\mathbf{P}$ (acting on the left), $P_\sigma = \frac{1}{2}(1 + \sigma_1 \cdot \sigma_2)$ is the spin-exchange operator and $\rho = \rho_n + \rho_p$ is the total nucleon density. We discuss in what follows only a simplified form in which the velocity-dependent terms are dropped.

2.2.2 Hartree-Fock for uniform nuclear matter with a simplified Skyrme force

Electron scattering experiments, together with other direct and indirect evidences, suggest that the inner part of nuclei (except perhaps the lightest ones) is characterised by an approximately constant density, whose value is about $\rho_0 \approx 0.16 \text{ fm}^{-3}$; in this region the proton and neutron densities do not differ too much. This suggests that an hypothetic extended system of equal number of protons and neutrons can be imagined, whose equilibrium density is ρ_0 with a corresponding energy given by the volume term of the mass formula, that is, $\approx -16 \text{ MeV}$ (in a uniform system with equal number of protons and neutrons there is no surface and no asymmetry, and the Coulomb force between protons must be of course neglected to make the system stable). Let us review some features of this hypothetical system and show that it can be studied with a simplification of the interaction given in (2.26).

In a uniform system of independent particles (Fermi gas) the Fermi momentum is given by¹

$$k_F = \left(\frac{3\pi^2 \rho}{2} \right)^{1/3} \approx 1.33 \text{ fm}^{-1}. \quad (2.27)$$

The corresponding Fermi energy is

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} \approx 37 \text{ MeV}, \quad (2.28)$$

the velocity as a fraction of the light velocity is

$$\frac{v}{c} = \frac{\hbar k_F}{mc} \approx 0.28, \quad (2.29)$$

¹Note that it has been considered that for each value of the momentum one can have four particles, two protons and two neutrons, each of the two having either spin up or down. This makes the factor in the next equation different from that already seen in the electronic case.

and the typical time to cross a nucleus with mass around ≈ 100 is

$$\tau = \frac{r_0 A^{1/3}}{v_F} \approx 6.6 \cdot 10^{-23} \text{ s}. \quad (2.30)$$

It can be shown that the typical mean free path of a nucleon is of the order, or larger, than the nuclear radius (cf. below, when the imaginary part of the optical potential is discussed); then, the typical time between two NN collisions is often estimated as $\tau \approx 10^{-23} - 10^{-22}$ s.

The wave functions in a uniform system are plane waves due to the translational invariance. This simplifies the calculation of the total energy (2.25), in keeping with the well-known replacements

$$\begin{aligned} \phi_{\mathbf{k}} &= \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k} \cdot \mathbf{r}} \\ \sum_i &\rightarrow \frac{g\Omega}{(2\pi)^3} \int d^3k, \end{aligned} \quad (2.31)$$

where g is the degeneracy equal to 4 and Ω is the quantisation volume. The kinetic part is

$$\begin{aligned} \sum_i \langle i|T|i \rangle &= \frac{4\Omega}{8\pi^3} \frac{1}{\Omega} \int d^3k \int d^3r e^{-i\mathbf{k} \cdot \mathbf{r}} \left(-\frac{\hbar^2}{2m} \nabla^2 \right) e^{i\mathbf{k} \cdot \mathbf{r}} \\ &= \frac{2}{\pi^2} \frac{\hbar^2}{2m} \frac{k_F^5}{5} \Omega. \end{aligned} \quad (2.32)$$

If we now re-express this quantity in terms of the density through Eq. (2.27), we obtain

$$\frac{T}{A} = \frac{3}{5} \left(\frac{3\pi^2}{2} \right)^{2/3} \frac{\hbar^2}{2m} \rho^{2/3}. \quad (2.33)$$

We now evaluate the potential energy with the simple force

$$v(\mathbf{r}, \mathbf{r}') = t_0(1 + x_0 P_\sigma) \delta(\mathbf{r}) + \frac{1}{6} t_3(1 + x_3 P_\sigma) \rho^\alpha(\mathbf{R}) \delta(\mathbf{r}). \quad (2.34)$$

For the antisymmetrized matrix elements of a zero-range force we can use the following simple trick,

$$\langle ij|v|kl \rangle_{\text{AS}} = \langle ij|v(1 - P_{12})|kl \rangle \quad (2.35)$$

where P_{12} is the operator that exchanges particles 1 and 2, given by the product $P_r P_\sigma P_\tau$ (being the three operators those exchanging position, spin and isospin respectively). For a simple δ -force like (2.34) P_r is 1. Since P_σ is known to be $\frac{1+\sigma_1\sigma_2}{2}$ and P_τ has a similar form, we can write

$$\begin{aligned}
v(1 - P_{12}) &= t_0(1 + x_0 P_\sigma)(1 - P_\sigma P_\tau)\delta(\mathbf{r}) \\
&+ \frac{1}{6}t_3(1 + x_3 P_\sigma)(1 - P_\sigma P_\tau)\rho^\alpha(\mathbf{R})\delta(\mathbf{r}) \\
&= t_0\delta(\mathbf{r})\left(\frac{3}{4} + \frac{x_0}{2}\sigma_1\sigma_2 - \frac{1}{2}\sigma_1\sigma_2 - \frac{x_0}{2}\tau_1\tau_2 - \frac{1}{2}\tau_1\tau_2 - \frac{1}{4}\sigma_1\sigma_2\tau_1\tau_2\right) \\
&+ \text{analogous term in } t_3.
\end{aligned} \tag{2.36}$$

It can be easily shown that the terms in $\sigma_1\sigma_2$ and $\tau_1\tau_2$ give a vanishing contribution in symmetric nuclear matter². Thus,

$$\begin{aligned}
\frac{1}{2}\sum_{ij}\langle ij|v|ij\rangle_{\text{AS}} &= \frac{1}{2}\sum_{ij}\langle ij|\frac{3}{4}\left(t_0 + \frac{t_3}{6}\rho^\alpha\right)\delta(\mathbf{r})|ij\rangle \\
&= \frac{1}{2}\frac{\Omega^2}{4\pi^6}\frac{1}{\Omega^2}\int d^3k d^3k' \int d^3r d^3r' e^{-i\mathbf{k}\cdot\mathbf{r}}e^{-i\mathbf{k}'\cdot\mathbf{r}'}\left(t_0 + \frac{t_3}{6}\rho^\alpha\right)\delta(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'} \\
&= \frac{3}{8}t_0\Omega\rho^2 + \frac{1}{16}t_3\Omega\rho^{\alpha+2}.
\end{aligned} \tag{2.37}$$

Therefore, the total energy per particle reads

$$\frac{E}{A} = \frac{3}{5}\left(\frac{3\pi^2}{2}\right)^{2/3}\frac{\hbar^2}{2m}\rho^{2/3} + \frac{3}{8}t_0\rho + \frac{1}{16}t_3\rho^{\alpha+1}. \tag{2.38}$$

If t_0 is positive and mimicks the main part of the NN attractive central force and t_3 is negative and associated with the repulsion that arises if the nucleons are too close to one another, it is clear that the energy per particle can have a minimum. By imposing that this minimum appears for $\rho_0 = 0.16 \text{ fm}^3$ and that E/A has the value -16 MeV at the minimum, one can fit the parameters t_0 and t_3 given an ansatz for the exponent α .

²The formal proof can be made available, but the basic argument is that the sum over ij in (2.25) involves with the same weight states with spin (isospin) up and down.

2.3 Nuclear superfluidity and BCS theory

2.3.1 Introduction

At the end of Sec. 2.2 we have stated that the HF theory, supplemented by a well calibrated effective interaction, is able to reproduce well the total binding energies and charge radii, that are the main properties of the nuclear ground states. However, this is true only for magic isotopes. It is mandatory, for open shell nuclei, to take into account at least the so-called “pairing” residual interaction within the framework of the Bardeen-Cooper-Schrieffer (BCS) or Bogoliubov approximations.

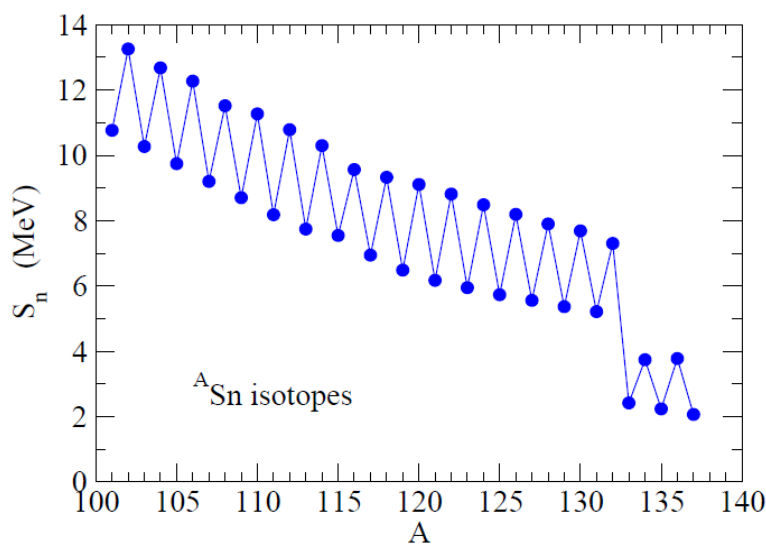


Figure 2.2: Separation energy of the Sn isotopes. We remind that $S_n = BE(N+1) - BE(N)$.

There are many proofs of the existence of pairing in nuclei.

- One is the trend of the binding energies themselves. Even in the simplest formula, the semi-empirical one, it is known that there is a pairing term whose value is $+\Delta$ for even-even nuclei, zero for odd nuclei, and $-\Delta$ for odd-odd nuclei. Thus, we must introduce the idea that nuclei get extra binding if the nucleons of the same kind can form pairs. This fact is clearly illustrated in Fig. 2.2. A rough estimate of Δ is $12/\sqrt{A}$

in MeV, but this is not so reliable for light systems while it works for medium-heavy ones.

- Historically, however, the previous one has not been the main push to introduce a special residual force that “pairs” nucleons. It has been noted by A. Bohr, B.R. Mottelson and D. Pines that in the mass regions where nuclei obey well the mean-field predictions, the first excited state of odd isotopes has quite low energy while that of even-even ones is around 1 MeV. This “gap” in the excitation spectrum is reminiscent of that already noted in the superconductors; then, the same BCS theory that has successfully explained ordinary superconductivity has been extended to atomic nuclei. Since the nucleon pairs are made up either with protons or with neutrons, we better speak of nuclear superfluidity rather than superconductivity in the present context.
- There are other evidences for nuclear pairing: for instance, as we shall mention below, the moment of inertia of deformed nuclei is a strong one.

In the nuclear BCS theory for systems with an even number of particles, the ansatz is that the pairs are made up with nucleons that are in the same orbital and can be coupled to total angular momentum $J = 0$. The state $|JM = 00\rangle$ can be formed as

$$\begin{aligned} |JM = 00\rangle &= \sum_{m_1 m_2} \langle j m_1 j m_2 | 00 \rangle |j m_1\rangle |j m_2\rangle \\ &= \sum_m \langle j m j - m | 00 \rangle |j m\rangle |j - m\rangle, \end{aligned} \quad (2.39)$$

where the first equality is the standard definition of a coupled state, and the second one stems from the fact that the Clebsch-Gordan coefficients vanish if $m_1 + m_2$ is not equal to the total M . Thus, these nucleons have the same quantum numbers n, l, j and, respectively, m and $-m$. We will omit, then, the labels n and l in what follows.

One could introduce at this stage the time-reversal operator $\hat{\tau}$ that changes t into $-t$. We remind that for a particle without spin this operator coincides with the complex conjugation³. Spin introduces extra complications that are

³This is easy to deduce since in the time-dependent Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \phi = H\phi,$$

dealt with in standard textbooks. Here we just state without proof the result of the application of the time-reversal operator to a state $|jm\rangle$, that is,

$$\hat{\tau}|jm\rangle \equiv |\widetilde{jm}\rangle = (-)^{j+m}|j-m\rangle. \quad (2.40)$$

Then, we can say that the states having m and $-m$ are also time-reversed states, aside from the phase that appears in the last equation.

In brief, in the BCS ansatz we introduce in the nuclear wave function terms that correspond to pairs of nucleons in time-reversed states, or that can be coupled to $J = 0$. The BCS ansatz for the total wave function is

$$|\text{BCS}\rangle = \Pi_{k>0} \left(u_k + v_k a_k^\dagger a_{\bar{k}}^\dagger \right) |-\rangle. \quad (2.41)$$

Here, k has replaced jm , $k > 0$ labels the states with positive projection of the total angular momentum, that is, $m > 0$, u and v are parameters of this wave function and $a^\dagger$ labels creation operators that act on the particle vacuum $|-\rangle$. The following comments are in order.

- The BCS wave function (2.41) generalises the HF one. In fact, for a nucleus with an even number of nucleons of each type, HF corresponds to filling orbitals with both m and $-m$ up to the Fermi energy. This is obtained from (2.41) if we set $v_k = 1$ and $u_k = 0$ when the state k lies below the Fermi energy, and we set $v_k = 0$ and $u_k = 1$ otherwise:

$$|\text{HF}\rangle = \Pi_{k < \text{Fermi level}} a_k^\dagger a_{\bar{k}}^\dagger |-\rangle. \quad (2.42)$$

In other words, if we use the variational wave function (2.41) we generalise the HF ansatz without excluding it *a priori*.

- The BCS wave functions contains many more terms, as compared to HF, in which pairs of nucleons can be scattered to unoccupied levels above the Fermi one. In the case in which the BCS state is not markedly different from the HF one, we can think of few particles scattered from just below to just above the Fermi level.
- One of the main features of the BCS wave function is that, in principle, it mixes states with different number of particles.

if H is Hermitean then, with $t \rightarrow -t$, ϕ^* becomes the solution.

2.3.2 Derivation of the BCS equations

The equations can be derived from the variational principle, as in the HF case, but using the more general wave function (2.41). To this aim, we need to write the Hamiltonian in second quantization as

$$H = T + V = \sum_{k_1 k_2} \langle k_1 | t | k_2 \rangle a_{k_1}^\dagger a_{k_2} + \frac{1}{4} \sum_{k_1 k_2 k_3 k_4} \langle k_1 k_2 | \bar{v} | k_3 k_4 \rangle a_{k_1}^\dagger a_{k_2}^\dagger a_{k_4} a_{k_3}, \quad (2.43)$$

where the notation is the same as in Sec. 2.2. Since the BCS wave function mixes states with different particle number, and we wish such number be conserved at least in average, we shall impose the variational principle by searching the minimum of $\langle \text{BCS} | H - \lambda N | \text{BCS} \rangle$, where λ is a Lagrange multiplier. In this case, the meaning of λ will be provided by the equation

$$\lambda \equiv \frac{\partial}{\partial N} \langle \text{BCS} | H | \text{BCS} \rangle. \quad (2.44)$$

In other words, λ represents the energy which is needed to add or remove a particle, that is, the Fermi energy. Thus, we have many variational parameters, that are the single-particle states k with their energies ε_k , as well as u_k , v_k and the Fermi energy λ .

We first show that the state $|\text{BCS}\rangle$ is normalized.

$$\langle \text{BCS} | \text{BCS} \rangle = \Pi_{k>0} \langle - | (u_k + v_k a_{\bar{k}} a_k) (u_k + v_k a_k^\dagger a_{\bar{k}}^\dagger) | - \rangle. \quad (2.45)$$

The only non-vanishing terms in vacuum expectation values are those in which the number of creation and annihilation operators is the same, so that contractions of the type $\overline{aa^\dagger}$ can provide a finite number. Thus, we are left with

$$\langle \text{BCS} | \text{BCS} \rangle = \Pi_{k>0} \langle - | u_k^2 + v_k^2 \overline{a_{\bar{k}} a_k a_k^\dagger a_{\bar{k}}^\dagger} | - \rangle = \Pi_{k>0} (u_k^2 + v_k^2). \quad (2.46)$$

This shows that we have to require

$$u_k^2 + v_k^2 = 1, \quad (2.47)$$

and this latter equation, valid for each k , supports the interpretation that u_k is the probability amplitude that the state k is empty, while v_k is the probability amplitude that the state k is occupied. In other terms, v_k^2 is the BCS occupation factor of the state k .

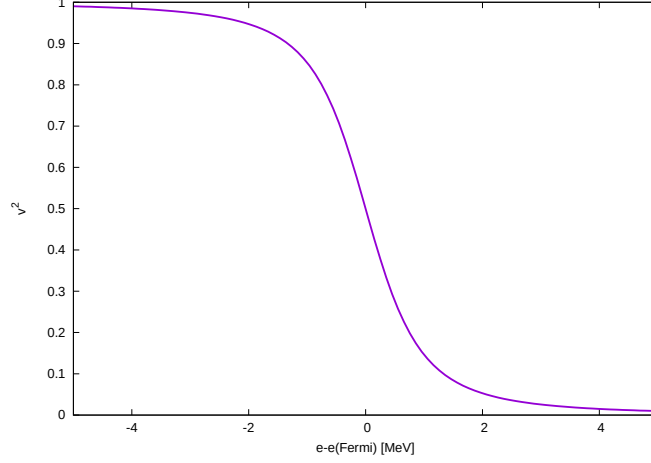


Figure 2.3: Schematic picture of the occupation factors v^2 as a function of $\varepsilon - \lambda$ [see also Eq. (2.62)].

We now calculate the quantity that will be minimised, namely

$$\langle \text{BCS} | H - \lambda N | \text{BCS} \rangle = \langle \text{BCS} | T - \lambda N | \text{BCS} \rangle + \langle \text{BCS} | V | \text{BCS} \rangle. \quad (2.48)$$

The first quantity at the r.h.s. includes one-body operators and the second is associated with the two-body operator V . The former is straightforward to evaluate. We use the standard definition of the number operator

$$N = \sum_k a_k^\dagger a_k, \quad (2.49)$$

and the result provided by Eq. (2.75), and we obtain

$$\begin{aligned} \langle \text{BCS} | T - \lambda N | \text{BCS} \rangle &= \sum_{k_1 k_2} (\langle k_1 | t | k_2 \rangle - \lambda \delta(k_1, k_2)) \langle \text{BCS} | a_{k_1}^\dagger a_{k_2} | \text{BCS} \rangle \\ &= \sum_k (\langle k | t | k \rangle - \lambda) v_k^2. \end{aligned} \quad (2.50)$$

Before we move to the expectation value of V , we must notice that the previous formula also provides the expectation value of the number operator.

One of the requirements of the BCS theory, as we have already mentioned, is that this expectation value is equal to the actual number of particles⁴. This amounts to

$$N = \sum_k v_k^2 = 2 \sum_{k>0} v_k^2. \quad (2.51)$$

The expectation value of V is also calculated in the appendix [cf. Eq. (2.77)]. If we sum the result of this equation to the one of (2.50), we obtain

$$\begin{aligned} \langle \text{BCS} | H - \lambda N | \text{BCS} \rangle &= \sum_k \left[(t_{kk} - \lambda) v_k^2 + \frac{1}{2} \sum_{k'} \bar{v}_{kk'kk'} v_k^2 v_{k'}^2 \right] \\ &+ \sum_{kk'>0} \bar{v}_{k\bar{k}k'\bar{k}'} u_k v_k u_{k'} v_{k'}. \end{aligned} \quad (2.52)$$

We now calculate

$$\delta \langle \text{BCS} | H - \lambda N | \text{BCS} \rangle = 0. \quad (2.53)$$

We use v_k as a variational parameter, that is, we equate the total derivative with respect to v_k to zero. Since $u_k = \sqrt{1 - v_k^2}$, then

$$\frac{d}{dv_k} = \frac{\partial}{\partial v_k} - \frac{v_k}{u_k} \frac{\partial}{\partial u_k}.$$

We apply this operator to (2.52) and, by multiplying by $\frac{u_k}{2}$ we obtain

$$\begin{aligned} 0 &= u_k v_k \left[(t_{kk} + t_{\bar{k}\bar{k}} - 2\lambda) + \sum_{k'} (\bar{v}_{kk'kk'} + \bar{v}_{\bar{k}k'\bar{k}k'}) v_{k'}^2 \right] \\ &+ \sum_{k'>0} \bar{v}_{k\bar{k}k'\bar{k}'} (u_k^2 - v_k^2) u_{k'} v_{k'}. \end{aligned} \quad (2.54)$$

We introduce the quantity

$$\tilde{\varepsilon}_k = \frac{1}{2} \left[(t_{kk} + t_{\bar{k}\bar{k}} - 2\lambda) + \sum_{k'} (\bar{v}_{kk'kk'} + \bar{v}_{\bar{k}k'\bar{k}k'}) v_{k'}^2 \right]. \quad (2.55)$$

⁴Note that we use, for the sake of simplicity, the same symbol for the particle number operator and the number of particles in the system. The context should make it clear what we precisely refer to.

Another important definition is that of the pairing gap,

$$\Delta_k = - \sum_{k' > 0} \bar{v}_{k\bar{k}k'\bar{k}'} u_{k'} v_{k'}. \quad (2.56)$$

With these definitions, Eq. (2.54) becomes

$$\begin{aligned} 0 &= 2u_k v_k \tilde{\varepsilon}_k + \Delta_k (v_k^2 - u_k^2), \\ 4\tilde{\varepsilon}_k^2 u_k^2 v_k^2 &= \Delta_k^2 (u_k^2 - v_k^2)^2. \end{aligned} \quad (2.57)$$

By using Eq. (2.47),

$$\begin{aligned} (u_k^2 + v_k^2)^2 &= 1 = u_k^4 + v_k^4 + 2u_k^2 v_k^2, \\ u_k^4 + v_k^4 &= 1 - 2u_k^2 v_k^2, \\ (u_k^2 - v_k^2)^2 &= 1 - 4u_k^2 v_k^2, \end{aligned} \quad (2.58)$$

one arrives at

$$4\tilde{\varepsilon}_k^2 u_k^2 v_k^2 = \Delta_k^2 - \Delta_k^2 u_k^2 v_k^2. \quad (2.59)$$

We introduce at this point the quasi-particle energy

$$E_k = \sqrt{\tilde{\varepsilon}_k^2 + \Delta_k^2}. \quad (2.60)$$

If this definition, together with the condition (2.47), are inserted into Eq. (2.59), an equation for v_k^2 is obtained, whose solution is straightforward and reads

$$v_k^2 = \frac{1}{2} \pm \frac{|\tilde{\varepsilon}_k|}{2E_k}. \quad (2.61)$$

It is easy to check that the solution that leads to the correct HF limit, in which v_k^2 should be 0 (1) above (below) the Fermi energy λ is

$$v_k^2 = \frac{1}{2} - \frac{\tilde{\varepsilon}_k}{2E_k}. \quad (2.62)$$

The corresponding solution for u_k^2 that has the correct HF limit, that is, 0 (1) below (above) the Fermi energy λ is

$$u_k^2 = \frac{1}{2} + \frac{\tilde{\varepsilon}_k}{2E_k}. \quad (2.63)$$

If we now write the product $u_{k'}v_{k'}$ from the two latter equations (2.62) and (2.63), we insert this in the definition (2.56), and we also use the other definition (2.60), we arrive at the gap equation

$$\Delta_k = - \sum_{k' > 0} \frac{\Delta_{k'}}{2E_{k'}} v_{k\bar{k}k'\bar{k}'} \quad (2.64)$$

The system made up with Eqs. (2.51,2.55,2.60,2.62,2.63,2.64) is closed. The equations are displayed within a box for the reader's convenience.

Before we attempt some schematic solutions of this system, we should give a tentative interpretation of the quantities we have introduced.

- The number equation (2.51) expresses the fact that, on average, the particle number is conserved. In fact, as we have already remarked, we ought to consider v_k^2 as the probability the level k is occupied or, equivalently, the average number of particles on the level k .
- The quantity $\tilde{\varepsilon}_k$ that has been introduced through Eq. (2.55) generalises the single-particle concept that has been introduced in the HF theory. In fact, it represents the difference between the single-particle energy and the Fermi energy λ . This can be shown in the following way. If we assume that the nucleus at hand is invariant under time-reversal, and we set ourselves in the HF limit in which $v_{k'}$ is 1 below the Fermi energy, then Eq. (2.55) becomes

$$\tilde{\varepsilon}_k = t_{kk} - \lambda + \sum_{k'} \bar{v}_{kk'kk'}. \quad (2.65)$$

The same equation can be derived if one subtracts λ from the single-particle energy ε_i deduced from Eq. (2.21). This deduction is made by multiplying Eq. (2.21) at left by $\varphi_i^*(\vec{r})$ and by integrating over $\vec{r}$.

- The quasi-particle definition (2.60) again reduces to a single-particle energy measured with respect to the Fermi energy λ in the HF limit. In the non-trivial case, however, we notice that the energy needed to add a nucleon to the system is larger than Δ_k . This is true at least if we have a system with an even number of particles and explains the gap in the spectrum which is exactly Δ_k . If the system has an odd number of nucleons, the odd nucleon is added on top of the BCS wave

function without taking part in any Cooper pair, so that its energy can be arbitrarily small with respect to the Fermi energy and the odd system does not display any gap. We have also explained, in this way, the pairing term $\approx 12/\sqrt{A}$ in the semi-empirical mass formula.

- The equation that determines whether the system has the simple HF solution or the non-trivial solution that we shall call superfluid from now on, is the gap equation (2.64). Intuitively, one can understand that if the density of levels is not extremely large, so that $\varepsilon_k - \lambda$ must remain finite, and $v_{k\bar{k}k'\bar{k}'}$ is small, the HF solution will be the right one. Otherwise, a superfluid solution will be preferred. We shall discuss this on the explicit examples that follows.

2.3.3 Solution of the BCS equations in schematic cases

In both the cases that we discuss below, we assume that the matrix elements that enter the gap equation are constant, namely we approximate $\bar{v}_{k\bar{k}k'\bar{k}'}$ with $-G$. In this case, it is convenient to start from the gap equation in the form (2.56),

$$\Delta_k = G \sum_{k'} u_{k'} v_{k'} = \Delta, \quad (2.66)$$

since this equation shows that the pairing gap becomes state-independent.

Solution for equally spaced levels

We consider two-fold degenerate levels having a constant spacing d , in an energy window which is symmetric with respect to the Fermi energy λ , so that the lowest level is at $\lambda - \Lambda$ and the upper level at $\lambda + \Lambda$. The density of levels is $g = 2/d$, and the density of levels with positive projection of the angular momentum is $g/2 = 1/d$. By approximating the sum in the gap equation (2.64) with an integral, we obtain

$$\begin{aligned} \Delta &= \frac{G}{2} \Delta \sum_{k' > 0, \lambda - \Lambda < \varepsilon_{k'} < \lambda + \Lambda} \frac{1}{\sqrt{(\varepsilon_{k'} - \lambda)^2 + \Delta^2}}, \\ 1 &= \frac{G}{2} \frac{g}{2} \int_{-\Lambda}^{+\Lambda} d\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta^2}}. \end{aligned} \quad (2.67)$$

The integral can be solved and its result is $2\sinh^{-1}\left(\frac{\Lambda}{\Delta}\right)$. The solution of the model is therefore

$$\Delta = \frac{\Lambda}{\sinh \frac{2}{gG}} \approx 2\Lambda e^{-2/Gg}. \quad (2.68)$$

In this solvable model one can see very clearly that the parameter that plays a role is the product of the typical value of the pairing matrix elements G times the density of levels. If this parameter is small, the gap essentially vanishes. The gap is finite if the pairing matrix elements are comparable with, or non negligible with respect to, the level spacing.

Solution for a single j -shell

Let us now make the further assumption that we restrict to a single level, that have degeneracy $2j + 1 \equiv 2\Omega$ (Ω is the number of states with positive projection of the angular momentum). The number equation becomes simply

$$N = 2 \sum_{k>0} v_k^2 = 2\Omega v^2, \quad (2.69)$$

so that

$$v = \sqrt{\frac{N}{2\Omega}} \quad u = \sqrt{1 - \frac{N}{2\Omega}}, \quad (2.70)$$

so that the pairing gap extracted from Eq. (2.66) becomes

$$\Delta = G\Omega \sqrt{\frac{N}{2\Omega} \left(1 - \frac{N}{2\Omega}\right)} = G\sqrt{\frac{N}{2} \left(\Omega - \frac{N}{2}\right)}. \quad (2.71)$$

This kind of trend like that of a reversed parabola, with a maximum when the shell is half-occupied ($N = \Omega$) and minima at zero in correspondence of the completely empty or filled shell, is typical also of microscopic calculations in the case of the isotopes $^{120-132}\text{Sn}$, where the single $h_{11/2}$ shell plays a major role.

2.A Appendices

2.A.1 Matrix elements of use for the BCS theory

We wish to calculate

$$\langle \text{BCS} | a_{k_1}^\dagger a_{k_2} | \text{BCS} \rangle. \quad (2.72)$$

A useful consideration to be made is that the BCS wave functions at the left and at the right will bring in, respectively, pairs of operators like $a_{\tilde{k}_1} a_{k_1}$ and $a_{k_2}^\dagger a_{\tilde{k}_2}^\dagger$. There is no way to obtain an even number of creation and annihilation operators with the same label, unless $k_1 = k_2$. Thus, we are left with

$$\begin{aligned} \langle \text{BCS} | a_k^\dagger a_k | \text{BCS} \rangle &= \\ &= \Pi_{l>0, l \neq k} \langle - | (u_l + v_l a_{\tilde{l}} a_l) (u_k + v_k a_{\tilde{k}} a_k) a_k^\dagger a_k (u_k + v_k a_k^\dagger a_{\tilde{k}}^\dagger) (u_l + v_l a_l^\dagger a_{\tilde{l}}^\dagger) | - \rangle. \end{aligned} \quad (2.73)$$

We can move the operators carrying l and $\tilde{l}$ as labels on the left side by exploiting the anticommutation relations. If $k \neq l$, all operators anticommute and exchanging two of them produces a minus sign, so that an even number of exchanges keeps invariant the product of operators.

$$\begin{aligned} \langle \text{BCS} | a_k^\dagger a_k | \text{BCS} \rangle &= \\ &= \Pi_{l>0, l \neq k} \langle - | (u_l + v_l a_{\tilde{l}} a_l) (u_l + v_l a_l^\dagger a_{\tilde{l}}^\dagger) (u_k + v_k a_{\tilde{k}} a_k) a_k^\dagger a_k (u_k + v_k a_k^\dagger a_{\tilde{k}}^\dagger) | - \rangle = \\ &= \Pi_{l>0, l \neq k} (u_l^2 + v_l^2) \langle - | (u_k + v_k a_{\tilde{k}} a_k) a_k^\dagger a_k (u_k + v_k a_k^\dagger a_{\tilde{k}}^\dagger) | - \rangle = \\ &= v_k^2 \langle - | \overline{a_{\tilde{k}} a_k a_k^\dagger a_{\tilde{k}}^\dagger} | - \rangle = v_k^2. \end{aligned} \quad (2.74)$$

In conclusion,

$$\langle \text{BCS} | a_{k_1}^\dagger a_{k_2} | \text{BCS} \rangle = \delta(k_1, k_2) v_{k_1}^2. \quad (2.75)$$

In order to calculate the expectation value of V , we fix two indices k and k' and define for the sake of brevity

$$|\psi\rangle \equiv \Pi_{k'' \neq k, k'} (u_{k''} + v_{k''} a_{k''}^\dagger a_{\tilde{k}''}^\dagger) | - \rangle.$$

Then,

$$\begin{aligned} \langle \text{BCS} | V | \text{BCS} \rangle &= \frac{1}{4} \sum_{k_1 k_2 k_3 k_4} \langle \psi | (u_k + v_k a_{\tilde{k}} a_k) (u_{k'} + v_{k'} a_{\tilde{k}'} a_{k'}) \times \\ &\times a_{k_1}^\dagger a_{k_2}^\dagger a_{k_4} a_{k_3} (u_{k'} + v_{k'} a_{k'}^\dagger a_{\tilde{k}'}^\dagger) (u_k + v_k a_k^\dagger a_{\tilde{k}}^\dagger) | \psi \rangle. \end{aligned} \quad (2.76)$$

Note that $|\psi\rangle$ is equivalent to the vacuum for operators that carry the labels k and k' . There are potentially many terms but one must exclude those that do not give any contribution because the creation and annihilation operators cannot be matched. After a tedious enumeration one arrives at

$$\langle \text{BCS} | V | \text{BCS} \rangle = \frac{1}{2} \sum_{kk'} \bar{v}_{kk'kk'} v_k^2 v_{k'}^2 + \sum_{kk'>0} \bar{v}_{k\tilde{k}k'\tilde{k}'} u_k v_k u_{k'} v_{k'}. \quad (2.77)$$

Chapter 3

Spectroscopy of even-even nuclei and collective vibrations

3.1 Introduction

Understanding the pattern of excited nuclear states is in general a hard task. The lowest nuclear excitations can be as low as 11 keV (first excited state in ^{237}U) or as large as 6 MeV (first excited state in ^{16}O). Data are collected e.g. at <http://www.nndc.bnl.gov/>. The states are characterised by the good quantum numbers of the nuclear Hamiltonian, namely total angular momentum and parity J^π .

Many effects contribute to make nuclear spectroscopy a complicated domain: the global nuclear shape (spherical or deformed) is responsible for creating different excitation patterns, shell effects play a major role (in closed shell systems particles need to cross a major shell to be excited, while in open shell systems one can have extremely low energy states), and pairing cannot be neglected as one should have understood from Sec. 2.3. These effects are particularly evident in the low-lying sector of the nuclear excitation spectrum. As is well known, nuclei have separation energies of the order of $\approx 7\text{--}8$ MeV, if we stay within the stability valley. Below this energy we can observe discrete states, and γ transitions provide information not only about the energies of the states, but also about their character as we shall discuss below. This is the realm which is normally referred to when one talks about nuclear spectroscopy. Above the separation energy, nucleons are in the continuum region. The excitation spectra are not discrete any longer and only in

very specific cases one identifies narrow structures. In this region, one rather searches prominent states that emerge from the complicated background and carry interesting information about the nuclear properties, like the so-called giant resonances.

With this general background in mind, we shall organise the discussion as follows. We restrict ourselves to the case of even-even nuclei, where the ground state has $J^\pi = 0^+$. We first introduce the typical patterns of the low-lying excited states that allow distinguishing between spherical and deformed nuclei. We then consider the former case, the one of spherical nuclei, and discuss the phenomenology and the theory of their oscillations.

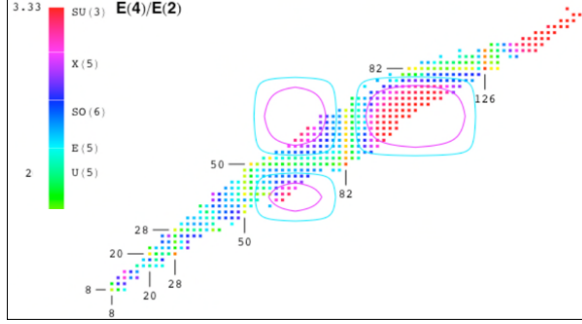
3.2 Vibrational and rotational nuclei

In Fig. 3.1, the quantity which is considered is

$$R = \frac{E(4_1^+)}{E(2_1^+)}, \quad (3.1)$$

where the energies are those of the first excited 4^+ and 2^+ states, respectively. Although such values tend to vary, there are certainly regions in which they are close either to 2 or 3.33. These values represent some paradigmatic nuclei whose excitation spectrum can be to a large extent interpreted in simple terms. We first note that the nuclei associated with $R = 2$ tend to lie on lines having constant Z or constant N , where either of them is a magic number. In the regions where neither of them is a magic number there are clear "islands" where nuclei associated with $R = 3.33$ tend to cluster.

A simple paradigm for a system is the harmonic oscillator spectrum. If the ground state is 0^+ , we can imagine that the excited states are made up of "phonons", or vibrational quanta, having a given angular momentum and parity. In many (although not all) even-even nuclei, the lowest of such states turns out to be a quadrupole, i.e. 2^+ phonon. In the harmonic case, we expect a two phonon state at twice the 2^+ energy. The two phonon state results from the coupling of 2^+ with itself, so we actually should expect a triplet of 0^+ , 2^+ and 4^+ states which are degenerate in the pure harmonic picture. Experimental spectra will inform us about the amount of anharmonicities. Thus, nuclei having R equal, or close, to 2, are spherical nuclei that can vibrate but in quantum mechanics cannot rotate.



Courtesy: P. Van Isacker

Figure 3.1: The ratio between the first excited 4^+ and 2^+ states is displayed along the whole nuclear chart, in the case of even-even nuclei. The labels close to the color code legend correspond to group symmetries that are not discussed in the present context.

Nuclei with a non-spherical shape can instead rotate. A classical rotor will display energies that scale like $J^2/2\mathcal{I}$ where J is the angular momentum and $\mathcal{I}$ is the moment of inertia. If we discard for the moment the question whether all values, even or odd, of the quantized angular momentum J are allowed, we find that for energies that scale like

$$E_{\text{rotational}} = \frac{J(J+1)}{2\mathcal{I}}, \quad (3.2)$$

the ratio R is exactly 3.33. Such nuclei are seen as deformed ones and the regions that are circled in Fig. 3.1 are usually referred to as "islands of deformation". They correspond to regions in which both neutrons and protons have open shells.

Our discussion leaves out the intermediate values of R . We will to some extent come back to them in the next Chapter after discussing nuclear deformation and rotation. We stick to spherical nuclei that vibrate in this Chapter.

3.3 Phenomenology of nuclear vibrations

There are vibrations characterised by different quantum numbers, the quadrupole one being one of the simplest and most likely to show up in nuclear spectra especially as first excited state. These quantum numbers are L (spatial angular momentum), S (spin angular momentum) and J (total angular momentum, given by $\vec{J} = \vec{L} + \vec{S}$). One more approximate quantum number is T (isospin). Generally speaking, spin excitations are higher in energy than non-spin ones, and $T = 1$ (isovector) excitations are higher than $T = 0$ (isoscalar) ones. In the discrete part of the spectrum, states that do not carry spin and isospin are prevalent.

We discuss, however, some of the states that lie above the particle threshold, that is, around ≈ 10 -30 MeV. The reason, as briefly mentioned above, is that in this energy region we find states that have resonant shape and emerge on the background of other excited states or, in other words, have large cross section associated with inelastic processes (we shall return to this point when discussing inelastic reactions).

The first evidence of the existence of such "Giant Resonances" has been found in experiments in which a beam of photons impinges on a nucleus. A large resonance has been found systematically in all nuclei, at an energy of the order of

$$E_{\text{IVGDR}} \approx 80 A^{-1/3}. \quad (3.3)$$

At such energies, it is easy to verify that the wavelength associated with the photons, $\lambda = \frac{2\pi\hbar c}{E} \approx 15 A^{1/3}$ fm, is much larger than the nuclear size. The protons feel mainly the electric field, that oscillates rapidly in time but is essentially uniform in space. In these conditions we are in the well-known dipole approximation. Protons are pulled apart from neutrons, but the strong proton-neutron attraction provides the restoring force. In the limit of a small external perturbation, we can model such situation as a forced harmonic motion. We expect a resonance when the external frequency matches the eigenmode of the system. We can describe this resonance in a classical fashion, but our aim is to relate its existence to the mean-field description of the previous Chapter, and this will be discussed below. The resonance is called Isovector Giant Dipole Resonance (IVGDR) because protons and neutrons move against each other.

We use this example to introduce some general tools that are very useful for the study of states like the Giant Resonances. If the external field (i.e.

the electromagnetic field in the above case) is written as $\hat{F}$, the probability amplitude to excite a state $|n\rangle$ is related to

$$\langle n|\hat{F}|0\rangle, \quad (3.4)$$

where $|0\rangle$ is the ground state (via, for instance, the Fermi golden rule). We define the strength function associated with that operator in the following way, namely

$$S(E) = \sum_n |\langle n|\hat{F}|0\rangle|^2 \delta(E - E_n). \quad (3.5)$$

Here, the sum is over a complete set of states that can be excited by the operator $\hat{F}$ and have energies E_n . This function carries clearly some dimension but the square of the matrix element of the operator $\hat{F}$ is associated with the probability of exciting the specific state $|n\rangle$. A state is dominant in the response to the external operator $\hat{F}$ if its contribution to strength function, $|\langle n|\hat{F}|0\rangle|^2$, is a large fraction of the total integral. The total integral is the non-energy weighted moment m_0 of S , namely

$$m_0 = \int dE S(E) = \sum_n |\langle n|\hat{F}|0\rangle|^2. \quad (3.6)$$

Other moments characterise the strength function, like the first or energy-weighted moment m_1 ,

$$m_1 = \int dE E S(E) = \sum_n E_n |\langle n|\hat{F}|0\rangle|^2. \quad (3.7)$$

A reasonable definition of the average energy that is transferred to the nucleus through the action of the operator $\hat{F}$ is

$$\bar{E} \equiv \frac{m_1}{m_0}. \quad (3.8)$$

The appearance of δ -functions in the strength function (3.5) is related to the assumption that the states are discrete. This is usually not the case because the excited states can have a finite lifetime due to decay to other states or particles or γ emission etc. In this case the matrix elements squared will be multiplied by functions having a width Γ , like gaussians or lorentzians. The width is related to the lifetime τ by

$$\Gamma = \frac{\hbar}{\tau}. \quad (3.9)$$

Giant resonances in the regions around tens of MeV have widths of the order of few MeV, that is, lifetimes of the order of few 10^{-22} s.

An external electromagnetic field excites the IVGDR in a practically exhaustive way. If one wishes to study other kinds of excitations one needs to resort to a strong, i.e. hadronic, field. The problem is then how to find a probe that excites the state of interest in a reasonable "exclusive" way. Most of the hadronic probes excite several states and disentangling them can be rather troublesome. Another issue is that we cannot avoid treating the reaction process that leads to the excitation. We will discuss inelastic scattering in the next Chapters, and we will touch upon the problem of how to relate the inelastic cross section to the strength function (3.5). Here we just mention that some types of experiments have been proven to be rather selective, and instrumental to identify specific giant resonances.

α -particle inelastic scattering excites only $S = 0$, $T = 0$ states. In the last decades, such experiments have been provided evidence of a $L = 2$, i.e. quadrupole, resonance at an energy

$$E_{\text{ISGQR}} \approx 65 \text{ A}^{-1/3}, \quad (3.10)$$

and of a $L = 0$, i.e. monopole, resonance at an energy

$$E_{\text{ISGMR}} \approx 80 \text{ A}^{-1/3}, \quad (3.11)$$

in medium-heavy nuclei. We shall discuss in the section concerning inelastic scattering how these, as well as other, resonances can be distinguished by looking at processes that have different scattering angles.

3.4 Linear response theory for nuclear vibrations

As stated previously, we work under the assumption that the external electromagnetic or hadronic field $\hat{F}$ is weak enough so that all equations can be computed at lowest order in such field (and in the quantities that characterise the nuclear response). In particular, we want to derive the response of a magic nucleus whose ground-state can be described in the HF approximation as described in Sec. 2.2.

Accordingly, we derive the linear response equations from the small amplitude limit of the *time-dependent* Hartree-Fock equations. The static HF

equations (2.21) read

$$h\phi_i = \varepsilon_i\phi_i, \quad (3.12)$$

and the single-particle HF Hamiltonian (sum of kinetic energy, Hartree and Fock potentials) is of course a function of the density. An external harmonic perturbation is supposed to act and to give a boost,

$$F(t) = f e^{-i\omega t} + f^\dagger e^{+i\omega t}; \quad (3.13)$$

then, the corresponding variations of ρ and h have the same form:

$$\delta\rho(t) = \delta\rho e^{-i\omega t} + \delta\rho^\dagger e^{+i\omega t}, \quad (3.14)$$

$$\delta h(t) = \delta h e^{-i\omega t} + \delta h^\dagger e^{+i\omega t}. \quad (3.15)$$

We write the equation of motion

$$i\hbar\dot{\rho} = [h, \rho] \quad (3.16)$$

and we solve it by assuming that $h = h^{(0)} + \delta h + f$, and that we can stop at linear terms either in $\delta\rho$, δh , or f :

$$i\hbar\dot{\rho} = [h^{(0)}, \delta\rho] + \left[\frac{\delta h}{\delta\rho}\delta\rho, \rho^{(0)}\right] + [f, \delta\rho]. \quad (3.17)$$

Note that in this latter equation we have used the fact that in the HF basis the Hamiltonian and the density are simultaneously diagonal, that is, $[\rho^{(0)}, h^{(0)}] = 0$. We express now Eq. (3.17) on a basis made up with particle-hole (p-h) excitations $|ph^{-1}\rangle$, by assuming that it is a complete basis. This is called Tamm-Dancoff approximation (TDA). Thus, we take the matrix elements of Eq. (3.17) between $\langle ph^{-1}|$ and the ground state $|0\rangle$; we thus obtain, for the terms that go like $e^{-i\omega t}$,

$$\hbar\omega \delta\rho_{ph} = \langle p|h^{(0)} \delta\rho - \delta\rho h^{(0)}|h\rangle + \langle p|\frac{\delta h}{\delta\rho}\delta\rho \rho^{(0)} - \rho^{(0)}\frac{\delta h}{\delta\rho}\delta\rho|h\rangle, \quad (3.18)$$

where $\delta\rho_{ph} = \langle ph^{-1}|\delta\rho|0\rangle$ and will be denoted by X_{ph} in the following. Note that, since we need to perform anyway the limit $f \rightarrow 0$ at the end, we have already neglected the commutator including the external field. We have also used the freedom to exchange the state $\langle h^{-1}|$ in the bra with the state $|h\rangle$ in the ket, and we shall use the equations $h^{(0)}|p\rangle = \varepsilon_p|p\rangle$, $h^{(0)}|h\rangle = \varepsilon_h|h\rangle$, whereas $\rho^{(0)}|p\rangle = 0$ and $\rho^{(0)}|h\rangle = |h\rangle$.

The last ingredient is the completeness relation

$$|0\rangle\langle 0| + \sum_{p'h'} |p'h'^{-1}\rangle\langle p'h'^{-1}| = 1, \quad (3.19)$$

that we insert it between the operators in the last matrix elements of the r.h.s. of the latter equation. We obtain

$$\hbar\omega \delta\rho_{ph} = (\varepsilon_p - \varepsilon_h) \delta\rho_{ph} + \sum_{p'h'} \langle ph^{-1} | \frac{\delta h}{\delta \rho} | p'h'^{-1} \rangle \delta\rho_{p'h'}. \quad (3.20)$$

By defining

$$A_{ph,p'h'} = \delta_{pp'} \delta_{hh'} (\varepsilon_p - \varepsilon_h) + \langle ph^{-1} | \frac{\delta h}{\delta \rho} | p'h'^{-1} \rangle, \quad (3.21)$$

and introducing the notation X_{ph} for the matrix elements $\langle ph^{-1} | \delta\rho | 0 \rangle$, we can write the TDA equations in matrix form:

$$AX = \hbar\omega X. \quad (3.22)$$

The meaning of the equation is quite straightforward. $\hbar\omega$ are the eigenvalues associated with a given vibrational mode. The first term in the matrix elements A is diagonal, and just expresses the fact that such eigenvalues would correspond to the unperturbed p-h energy difference $\varepsilon_p - \varepsilon_h$ in absence of any interaction. The second term in the matrix elements A couples such p-h excitations between one another and provides eigenstates that are mixed. It is easy to demonstrate that in simple cases (interactions V without density dependence) the matrix element of $\frac{\delta h}{\delta \rho}$ is just the antisymmetrized matrix element of the potential V . X_{ph} represents the amplitude, in the eigenmode, that is associated with the specific configuration $|ph^{-1}\rangle$. We may like to call collective those states in which several, or many, amplitudes X_{ph} are non negligible. However, the schematic model discussed in the following Section will be quite instrumental in showing that the relative phase of such amplitudes also matters. Fig. 3.2 is a cartoon of the TDA equations if the second term at l.h.s. is neglected.

A more sophisticated theory considers the possibility that the ground-state is more general than the HF one. In other words, p-h pairs may be present in a more exact ground-state. Then, the linear response formalism

may include the possibility that the p-h pairs are destroyed. In this case, the completeness relation becomes:

$$|0\rangle\langle 0| + \sum_{p'h'} |p'h'^{-1}\rangle\langle p'h'^{-1}| + |h'p'^{-1}\rangle\langle h'p'^{-1}| = 1. \quad (3.23)$$

If we include states of the type $|hp^{-1}\rangle$ in the basis, the derivation becomes less straightforward and will be omitted here. We just quote the final result, which is, in matrix form,

$$\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \hbar\omega \begin{pmatrix} X \\ Y \end{pmatrix}. \quad (3.24)$$

This is called Random Phase Approximation (RPA) equation. In it,

$$B_{ph,p'h'} = \langle ph^{-1} | \frac{\delta h}{\delta \rho} | h'p'^{-1} \rangle, \quad (3.25)$$

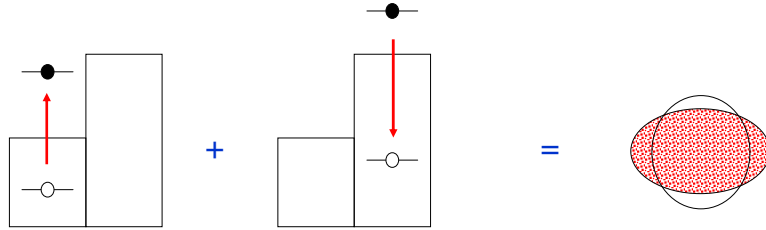


Figure 3.2: Schematic representation of the linear response equation. See the text for more discussion.

while Y_{ph} is the amplitude of $|hp^{-1}\rangle$ in the eigenmode, that is, $\langle hp^{-1}|\delta\rho|0\rangle$.

3.5 Solution of the TDA equations in a schematic model

We consider, as a simple model for e.g. the IVGDR, a schematic solution for the TDA matrix. We assume that we have N degenerate p-h excitations; in the IVGDR case for a magic nucleus, this corresponds to consider holes in the highest occupied shell, particles in the lowest unoccupied shell, and p-h energy differences all degenerate and equal to ε [according to Eq. (2.12) ε will be $\approx 41 \text{ A}^{-1/3}$]. We also take the matrix elements that appear in A all constant and equal to v , so that the TDA matrix becomes

$$\begin{pmatrix} \varepsilon + v & v & \dots & v \\ v & \varepsilon + v & \dots & v \\ \dots & & & \dots \\ v & v & \dots & \varepsilon + v \end{pmatrix}. \quad (3.26)$$

Such simple model is exactly solvable.

In the two by two case,

$$\begin{pmatrix} \varepsilon + v & v \\ v & \varepsilon + v \end{pmatrix}, \quad (3.27)$$

the eigenvalues and eigenvectors are

$$\begin{aligned} \hbar\omega &= \varepsilon, & X &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\ \hbar\omega &= \varepsilon + v & X &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \quad (3.28)$$

We may call the first state as "incoherent" because the two amplitudes are in opposition of phase while the second state may be said to be "coherent". The meaning of this label becomes more transparent if we calculate the the matrix element associated with the operator $\hat{F}$ as defined in Eq. (3.4), by assuming the X amplitudes are real,

$$\langle n|\hat{F}|0\rangle = \sum_{ph} X_{ph} \langle ph^{-1}|\hat{F}|0\rangle. \quad (3.29)$$

3.5. SOLUTION OF THE TDA EQUATIONS IN A SCHEMATIC MODEL 53

Around the Fermi surface, we expect that the matrix elements $\langle p|\hat{F}|h\rangle$ of a one-body operator are quite similar and can be approximated by a constant number M . This depends of course on the choice of the operator. For an electric or magnetic operator one expects to find these matrix elements are of the order of one Weisskopf, or single-particle, unit. Then, if one considers the "incoherent" state the transition amplitude associated with $\hat{F}$ tends to vanish, while its value is increased with respect to the single-particle value in the "coherent" case.

The matrix equation (3.26) can be solved in the general case with N p-h transitions. In that case, one finds there is only one coherent mode, having eigenvalue $\varepsilon + Nv$ and the corresponding eigenvector of the type

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}. \quad (3.30)$$

The transition amplitude associated with this state reads

$$\langle n|\hat{F}|0\rangle = \sum_{ph} X_{ph} \langle p|\hat{F}|h\rangle \approx N \frac{1}{\sqrt{N}} M = \sqrt{N} M. \quad (3.31)$$

Then, the quantity associated to the transition probability is the square,

$$|\langle n|\hat{F}|0\rangle|^2 = N M^2. \quad (3.32)$$

In the case of the IVGDR, we can imagine that the term Nv in the eigenvalue brings the excitation energy from the "unperturbed" value around 41 A^{-1/3} to the value 80 A^{-1/3} already mentioned in Eq. (3.3). The fact that this is a "giant" state stems from the last Eq. (3.32): its probability to be excited is N times larger the probability of a single-particle state. The situation is depicted in Fig. 3.3.

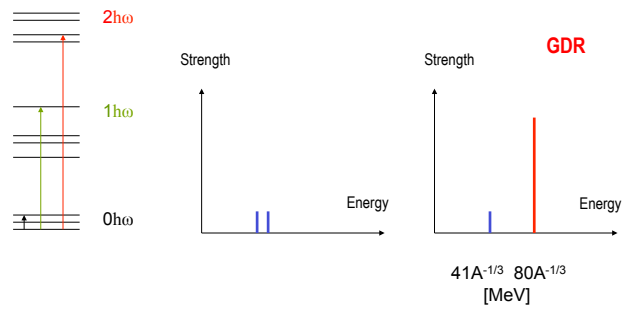


Figure 3.3: The emergence of the IVGDR in the schematic TDA model. See the text for the discussion.

Chapter 4

Nuclear deformation and related phenomena

4.1 Introduction to nuclear shapes

The usual way to parametrize the nuclear shapes, when they deviate from sphericity, is to define the radius as

$$R(\theta, \phi) = R_0 \left[1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \phi) \right], \quad (4.1)$$

in keeping with the fact that spherical harmonics are a complete set to describe scalar functions on a spherical surface. The parameters $\alpha_{\lambda\mu}$ are complex numbers, and they will be taken as small in what follows, in the sense that $|\alpha_{\lambda\mu}|^2 \ll |\alpha_{\lambda\mu}|$.

We would like to describe deformations that do not change the nuclear volume. We exclude the term $\lambda = 0$ from the sum in Eq. (4.1) to this aim, because $Y_{00}^* = \frac{1}{\sqrt{4\pi}}$ is a constant and the volume changes unless α_{00} vanishes. We actually need to exclude the terms with $\lambda = 1$ as well, the reason being that such terms describe a translation of the system¹. Then, the sum starts

¹One can show this with elementary geometry, in the case at hand in which $|\alpha_{\lambda\mu}|^2 \ll |\alpha_{\lambda\mu}|$.

from $\lambda = 2$ in Eq. (4.1). The volume of the system is

$$\begin{aligned}
 \int d\Omega \int_0^{R(\theta,\phi)} dR R^2 &= \int d\Omega \frac{R_0^3}{3} \left| 1 + \sum_{\lambda \geq 2, \mu} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \phi) \right|^3 = \\
 &= \int d\Omega \frac{R_0^3}{3} \left(1 + 3 \sum_{\lambda, \mu, \lambda', \mu'} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \phi) \alpha_{\lambda'\mu'}^* Y_{\lambda'\mu'}(\theta, \phi) \right) = \\
 &= \frac{4}{3} \pi R_0^3 \left(1 + 3 \sum_{\lambda\mu} |\alpha_{\lambda\mu}|^2 \right), \tag{4.2}
 \end{aligned}$$

where Ω is here a shorthand notation for the solid angle defined by θ and ϕ , and in the second equality we have stopped at the lowest order in the parameters α and we have used the orthonormality of the spherical harmonics,

$$\int d\Omega Y_{\lambda\mu}^*(\theta, \phi) Y_{\lambda'\mu'}(\theta, \phi) = \delta(\lambda, \lambda') \delta(\mu, \mu').$$

The result of Eq. (4.2) shows that if we neglect the square of the parameters $\alpha_{\lambda\mu}$ the volume is unchanged.

Although there is active research and some evidence of octupole deformation, static quadrupole deformation is more widespread in the isotope chart and more well established. We shall discuss it in what follows. Thus, we restrict to

$$R(\theta, \phi) = R_0 \left[1 + \sum_{\mu=-2}^2 \alpha_{2\mu} Y_{2\mu}^*(\theta, \phi) \right]. \tag{4.3}$$

4.2 Quadrupole deformation: intrinsic parameters and orientation in the laboratory system

We define two coordinate systems, the intrinsic one O' and the laboratory one O . The intrinsic system or intrinsic frame is defined in the following way: given a matter distribution that is (approximately) constant within the volume defined by the radius (4.3), we can diagonalize the tensor of inertia and identify the axes of the intrinsic system as those in which such tensor is diagonal. It is easy to convince oneself that such tensor has components

4.2. QUADRUPOLE DEFORMATION: INTRINSIC PARAMETERS AND ORIENTATION IN THE I

ij (where these indices assume the values x, y, z) that are proportional to α_{ij} . This quantity is the coefficient that weights $x_i x_j$ in the expansion (4.3) when this is written in cartesian coordinates. We avoid the straightforward yet lengthy steps associated with this transformation. It is possible to write Eq. (4.3) in cartesian coordinates,

$$R = R_0 \left(1 + \alpha_{xx} \frac{x^2}{r^2} + \dots + \alpha_{xy} \frac{xy}{r^2} + \dots \right), \quad (4.4)$$

use the well known coordinate transformations that allow expressing the spherical harmonics in cartesian coordinates, like

$$Y_{20} = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2 \theta - 1) = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{2z^2 - x^2 - y^2}{r^2}$$

and similar ones that can be found in standard textbooks, transform Eq. (4.3) accordingly and identify the coefficients that appear in Eq. (4.4). The result is

$$\begin{aligned} \alpha_{\pm 2} &= \sqrt{\frac{2\pi}{15}} (\alpha_{xx} - \alpha_{yy} \pm 2i\alpha_{xy}), \\ \alpha_{\pm 1} &= \mp 2 \sqrt{\frac{2\pi}{15}} (\alpha_{xz} \pm i\alpha_{yz}), \\ \alpha_0 &= \sqrt{\frac{\pi}{5}} \frac{2}{3} (\alpha_{zz} - \alpha_{xx} - \alpha_{yy}). \end{aligned} \quad (4.5)$$

We will label as a instead of α the coefficients in the intrinsic frame. The latter equations show that in the intrinsic frame in which the a 's in cartesian coordinates must form a diagonal tensor,

$$a_{\pm 1} = 0, \quad a_{-2} = a_2. \quad (4.6)$$

The latter equation shows that in the intrinsic frame there are only two independent parameters that define the shape. These are usually expressed in terms of β and γ defined as

$$a_0 = \beta \cos \gamma, \quad a_2 = \frac{\beta}{\sqrt{2}} \sin \gamma. \quad (4.7)$$

The laboratory system O is rotated with respect to O' . The rotation that brings O into O' , as any rotation in three dimensions, can be written in terms

of the three Euler angles $\{\phi, \theta, \psi\}$ that are denoted collectively as Ω^2 . The relationship between the spherical harmonics in the two systems read

$$Y_{\lambda\mu}(\theta', \phi') = \sum_{\mu'} \mathcal{D}_{\mu'\mu}^{(\lambda)}(\Omega) Y_{\lambda\mu'}(\theta, \phi), \quad (4.8)$$

where $\mathcal{D}$ are the Wigner D -functions. Note that we do not need the explicit expression of those functions for what follows. The latter equation translates the fact that angular momentum is conserved by any rotation while its z -component is not. A more general remark is that the three Euler angles and the parameters β and γ are equivalent to the five parameters $\alpha_{2\mu}$: there are five degrees of freedom associated with the quadrupole deformation but in the intrinsic frame we better understand that two of them are actually related to the nuclear shape, while the remaining three depend only on the orientation. In conclusion, in the intrinsic frame we can write

$$R(\theta', \phi') = R_0 [1 + a_0 Y_{20}(\theta', \phi') + 2a_2 Y_{22}(\theta', \phi')], \quad (4.9)$$

in keeping with the fact that Y_{20} is real and $Y_{2-2} = Y_{22}$ is also real (the relationship to be kept in mind is $Y_{\lambda\mu}^* = (-)^{\mu} Y_{\lambda-\mu}$). We shall drop the prime superscript in what follows for the sake of simplicity.

It is now easy to plug both the definition (4.7) and the explicit form of the spherical harmonics into Eq. (4.9), and look at the nuclear shape. The explicit result of such replacements reads

$$R(\theta, \phi) = R_0 \left\{ 1 + \sqrt{\frac{5}{16\pi}} \beta \left[\cos\gamma (3\cos^2\theta - 1) + \sqrt{3} \sin\gamma \sin^2\theta (\cos^2\phi - \sin^2\phi) \right] \right\}. \quad (4.10)$$

Specific expression can be easily found that define the variation $\delta R \equiv R - R_0$ along the three principal axes,

$$\begin{aligned} \delta R_x &= \sqrt{\frac{5}{4\pi}} \beta R_0 \cos\left(\gamma - \frac{2\pi}{3}\right), \\ \delta R_y &= \sqrt{\frac{5}{4\pi}} \beta R_0 \cos\left(\gamma + \frac{2\pi}{3}\right), \\ \delta R_z &= \sqrt{\frac{5}{4\pi}} \beta R_0 \cos\gamma. \end{aligned} \quad (4.11)$$

²The definition of Euler angles can be found in standard textbooks. Note that, although we use the same notation, one should not mix up the Euler angles with the polar angles θ, ϕ in the frame O .

4.3. ROTATIONAL BANDS OF A QUADRUPOLE, AXIALLY-DEFORMED EVEN-EVEN NUCLEI

In the case $\gamma = 0$, one finds that $\delta R_x = \delta R_y < 0$ while $\delta R_z > 0$. The nucleus is *axially symmetric* (z is the symmetry axis) and *prolate* (it is elongated along the symmetry axis and shrunk on the perpendicular plane). It can be shown that when γ deviates from 0 one has *triaxial shapes*, that is, shapes in which no axis remains a symmetric one against rotations. When γ reaches the value $\pi/3$, one can deduce from a simple calculation based again on Eq. (4.11) that $\delta R_x = \delta R_z > 0$ and $\delta R_y < 0$. This shape is still axially symmetric (this time, y is the symmetry axis) but it is shrunk around this symmetry axis and enlarged on the perpendicular plane. Such form is said to be *oblate*. Values of γ outside the interval $(0, \pi/3)$ produce the same shapes.

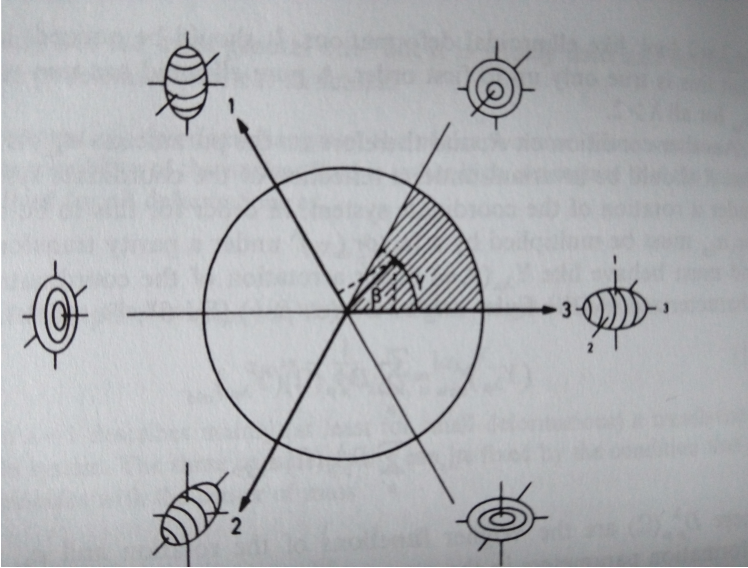


Figure 4.1: Schematic view of the nuclear shapes, taken from P. Ring and P. Schuck.

4.3 Rotational bands of a quadrupole, axially-deformed even-even nucleus

Deformed nuclei can rotate. We can start by making the adiabatic hypothesis, in which the rotational motion and the intrinsic motion of the nucleons within the range of the deformed potential happen on different time scales

and are decoupled. This hypothesis is well grounded in many medium-heavy nuclei, where the low-lying rotational states are in 10-100 keV range while the nucleon motion, as we have seen, is characterised by energies which are about two orders of magnitude larger (that is, associated time scales are two orders or magnitude smaller).

Within the adiabatic hypothesis, the Hamiltonian can thus be written as a sum

$$H = H_{\text{coll}} + H_{\text{intr}}, \quad (4.12)$$

where the collective Hamiltonian depends only on the coordinates of the rotation of the nucleus as a whole, that is, the three Euler angles Ω that describe the orientation of the intrinsic system with respect to the laboratory system. The intrinsic Hamiltonian depends instead on the nucleons coordinates in the intrinsic system, $\vec{r}_1 \dots \vec{r}_A$. In this hypothesis, the wave function factorizes

$$\psi = \mathcal{D}_{KM}^I(\Omega) \chi_K(\vec{r}_1 \dots \vec{r}_A). \quad (4.13)$$

We now discuss the meaning of the labels we have put, that ought to correspond to good quantum numbers of the system. We shall call $\vec{J}$ the angular momentum associated with the intrinsic degrees of freedom. If the system is not spherical in the intrinsic frame, but has a rotational symmetry around z' , the component of this angular momentum around z' is conserved: its eigenvalue is a good quantum number and we shall label it as Ω . We call $\vec{R}$ the angular momentum associated with the collective rotation, and $\vec{I}$ the total angular momentum, $\vec{I} = \vec{J} + \vec{R}$. The projections of this total angular momentum are also conserved. They are called respectively M (projection on z) and K (projection on z')³. In the specific case of an axially symmetric nucleus the rotation must take place around an axis which is perpendicular to the symmetry axis z' . Therefore, $R_{z'} = 0$ and K coincides with Ω . We shall restrict to this case and use K from now on. Fig. 4.2 depicts the quantities we have just defined.

The intrinsic wave function can, of course, only depend on K . The collective wave function $\mathcal{D}$ must be a Wigner function because mathematically

³This statement follows from the fact that the square of the total angular momentum I^2 commutes both with its z and z' projections. We omit here the formal proof of this fact, that is again based on the transformation from I_z to $I_{z'}$ via the Euler angles.

served reads

$$\psi_{IKM} = \sqrt{\frac{2I+1}{16\pi^2}}(1 + \delta_{K0}) [\mathcal{D}_{KM}^I(\Omega)\chi_K(\vec{r}'_1 \dots \vec{r}'_A) + (-)^{I+K}\mathcal{D}_{-KM}^I\chi_{-K}(\vec{r}'_1 \dots \vec{r}'_A)] . \quad (4.14)$$

If we consider even-even nuclei, K must be even. In the ground-state configuration $K = 0$. I must be larger than K . Eq. (4.14) shows that if $K = 0$, then I must be even. So, we expect that the *ground-state* rotational band is made up with states having $I^\pi = 0^+, 2^+, 4^+ \dots$. These states have energies

$$E_I = \frac{\hbar^2}{2\mathcal{I}}I(I+1), \quad (4.15)$$

that is, they form the typical spectrum of a rotor. From the measurements of the energies of the γ transitions, we can reconstruct such spectrum and (i) verify if it strictly obey the rotor scaling, but also (ii) extract values of the moments of inertia. In most of the nuclei with mass between ≈ 150 and ≈ 190 , and mass larger than ≈ 220 we find very regular rotational bands extending to high values of the angular momentum I .

4.4 Electromagnetic transitions and extraction of the intrinsic value of β

Not only the energies, but the electromagnetic transition probabilities among members of a rotational band can be measured. In the case that we have discussed in the last part of the previous Section, where the states have $I^\pi = 0^+, 2^+, 4^+ \dots$, the transitions⁴ are electric quadrupole ones, E2. The reduced transition probability $B(E2)$ between a state with initial (final) angular momentum I_i (I_f) reads

$$B(E2, I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \sum_{M_i M_f} |\langle I_f M_f | \hat{Q}_{2\mu} | I_i M_i \rangle|^2, \quad (4.16)$$

where the quadrupole operator could be written explicitly as $\sum_i e_i r_i^2 Y_{2\mu}(\hat{r}_i)$ and the sum runs over nucleons. This operator is obviously defined in the

⁴We assume familiarity with: definition of multipole electromagnetic transitions and of the associated reduced transition probability B , definition of single-particle units at least for the electric transitions.

4.4. ELECTROMAGNETIC TRANSITIONS AND EXCTRACTION OF THE INTRINISIC VALUE

laboratory frame O where the photon is detected but could be transformed in the intrinsic frame. We have learnt from the previous Sections that the spherical harmonics, as any tensor carrying good angular momentum, can be transformed from one frame to another by using the Wigner functions. Then,

$$\hat{Q}_{2\mu} = \sum_{\nu} \mathcal{D}_{\nu\mu}^2(\Omega) \hat{Q}_{2\nu}^{\text{intr}}, \quad (4.17)$$

and, by sticking to the case $K = 0$ and reading the wave functions from (4.14),

$$\langle I_f M_f | \hat{Q}_{2\mu} | I_i M_i \rangle = \frac{1}{8\pi^2} \sqrt{(2I_i + 1)(2I_f + 1)} \sum_{\nu} \int d\Omega \mathcal{D}_{0M_f}^{I_f*}(\Omega) \mathcal{D}_{\nu\mu}^2(\Omega) \mathcal{D}_{0M_i}^{I_i}(\Omega) \langle \chi_0 | \hat{Q}_{2\nu}^{\text{intr}} | \chi_0 \rangle, \quad (4.18)$$

where the bra and ket notation refer to the integral on the intrinsic coordinates $\vec{r}_1 \dots \vec{r}_A$ on which the intrinsic wave functions χ and the intrinsic operator $\hat{Q}^{\text{intr}}$ depend. The integral over the Wigner functions that appear in the latter equation is known,

$$\int d\Omega \mathcal{D}_{\mu_3 m_3}^{j_3*} \mathcal{D}_{\mu_2 m_2}^{j_2} \mathcal{D}_{\mu_1 m_1}^{j_1} = \frac{8\pi^2}{2j_3 + 1} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle \langle j_1 \mu_1 j_2 \mu_2 | j_3 \mu_3 \rangle.$$

Thus, the matrix elements (4.18) becomes

$$\langle I_f M_f | \hat{Q}_{2\mu} | I_i M_i \rangle = \sqrt{\frac{2I_i + 1}{2I_f + 1}} \langle I_i M_i 2\mu | I_f M_f \rangle \langle I_i 0 20 | I_f 0 \rangle \langle \chi_0 | \hat{Q}_{\nu=0}^{\text{intr}} | \chi_0 \rangle. \quad (4.19)$$

The second Clebsch-Gordan coefficients in this equation implies that ν must vanish⁵. Then, the value of the reduced transition probability reads

$$\begin{aligned} B(E2, I_i \rightarrow I_f) &= \frac{1}{2I_i + 1} |\langle \chi_0 | \hat{Q}_{20}^{\text{intr}} | \chi_0 \rangle|^2 \frac{(2I_i + 1) \langle I_i 0 20 | I_f 0 \rangle^2}{(2I_f + 1)} \sum_{M_i M_f} \langle I_i M_i 2\mu | I_f M_f \rangle^2 \\ &= \langle I_i 0 20 | I_f 0 \rangle^2 |\langle \chi_0 | \hat{Q}_{20}^{\text{intr}} | \chi_0 \rangle|^2. \end{aligned} \quad (4.20)$$

This equation expresses the very remarkable result that in a rotational band all transitions depend on a quantity which is related to the square of the

⁵We remind that for every non-zero coefficient $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$, the equality $m_1 + m_2 = m_3$ must hold.

intrinsic quadrupole moment, and the dependence on the specific initial and final states is a geometric, universal one - associated with the square of a Clebsch-Gordan coefficient.

We now show how the common quantity appearing in all reduced transition probability, the aforementioned intrinsic quadrupole moment, is related to β . We start from Eq. (4.10) restricted to $\gamma = 0$:

$$R(\theta) = R_0 \left[1 + \sqrt{\frac{5}{16\pi}} \beta (3\cos^2\theta - 1) \right]. \quad (4.21)$$

We assume that this is the radius associated both with the mass and charge distribution. The charge is assumed to be uniform within the volume defined by $R(\theta)$. We have shown in the first part of this Chapter [cf. Eq. (4.2)] that at the lowest order in β the volume is conserved, so that

$$\rho_{\text{charge}} = \frac{Z}{\frac{4}{3}\pi R_0^3}. \quad (4.22)$$

If we replace the square of the intrinsic wave function $|\chi|^2$ with the intrinsic density distribution, we obtain

$$\begin{aligned} \langle \chi_0 | \hat{Q}_{20}^{\text{intr}} | \chi_0 \rangle &= \int d^3r \, e \rho_{\text{charge}} r^2 Y_{20} = \frac{3Ze}{16\pi R_0^3} 2\pi \sqrt{\frac{5}{\pi}} \int_{-1}^{+1} d\mu (3\mu^2 - 1) \int_0^{R(\theta)} dr \, r^4 \\ &= \frac{3Ze}{8R_0^3} \sqrt{\frac{5}{\pi}} \int_{-1}^{+1} d\mu (3\mu^2 - 1) \frac{R_0^5}{5} \left[1 + \frac{5}{4} \sqrt{\frac{5}{\pi}} \beta (3\mu^2 - 1) + \dots \right] = \\ &= \frac{3ZeR_0^2}{4\pi} \beta + \dots \end{aligned} \quad (4.23)$$

where in the second step we have used the definition of

$$Y_{20} = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3\cos^2\theta - 1)$$

already given previously, and we have defined $\mu = \cos\theta$, and in the following steps we have retained the term at lowest order in β by marking the missing ones with dots. If β is small, and the approximation is meaningful, we find, by replacing the latter result (4.23) in Eq. (4.20):

$$B(E2, I_i \rightarrow I_f) = \langle I_i 0 2 0 | I_f 0 \rangle^2 \left(\frac{3ZeR_0^2}{4\pi} \right)^2 \beta^2. \quad (4.24)$$

In conclusion, the measurement of the reduced transition probability gives access to β^2 .

4.5 Moments of inertia

If the value of β can be extracted from the last equation, (4.24), and the value of the moment of inertia is known from the energies of the rotational spectrum, one can ask oneself the question whether the rotation has a rigid-body character. By assuming a mass distribution which is again uniform within the radius defined by Eq. (4.21), namely $\rho_{\text{mass}} = 3mA/4\pi R_0^3$, we can calculate the so-called rigid-body moment of inertia. In the case of an axially symmetric nucleus the rotation is around an axis perpendicular to the symmetry axis z' , like x' or y' . Of course, $\mathcal{I}_{x'} = \mathcal{I}_{y'} = \mathcal{I}$. The moment of inertia can be evaluated in a similar way as the one which led us to the intrinsic quadrupole moment in Eq. (4.23), namely

$$\mathcal{I} = \frac{2}{5}mAR_0^2 \left(1 + \frac{1}{2}\sqrt{\frac{5}{4\pi}}\beta \right). \quad (4.25)$$

With the values of β extracted from the electromagnetic transitions⁶ one obtains moment of inertia that are about a factor 2 larger than experiment. Superfluidity makes this value smaller.

4.6 Motion of a particle in a deformed potential: simple features

A simple model that describes the changes that the single-particle states undergo, when changing a spherical into a deformed shape, is that of the deformed harmonic oscillator. If ω_0 is the typical frequency that mimics the nuclear case, namely $\hbar\omega_0 \approx 41A^{-1/3}$ [cf. Eq. (2.12)], in an axially deformed nucleus we can assume that we have different frequencies along the symmetry axis z' (ω_z) and along the two perpendicular axes x' and y' ($\omega_\perp$). Thus, we can write

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m[\omega_\perp^2(x^2 + y^2) + \omega_z^2 z^2]. \quad (4.26)$$

The conservation of the volume can be expressed by the condition

$$\omega_\perp^2 \omega_z = \omega_0^3, \quad (4.27)$$

⁶Or from other experimental techniques, like laser atomic spectroscopy.

and one can easily verify that this condition is fulfilled by

$$\omega_z = \omega_0 \left(1 - \frac{2}{3}\epsilon\right), \quad \omega_\perp = \omega_0 \left(1 + \frac{1}{3}\epsilon\right) \quad (4.28)$$

at lowest order in ϵ , that is, by neglecting terms of the order ϵ^2 . The simple model of Eq. (4.26) can be improved by adding a spin-orbit term, and empirical terms that make it more similar to the Woods-Saxon shape. When these terms are added, one has the so-called Nilsson model.

One can rewrite Eq. (4.26) as

$$\begin{aligned} H &= H_0 + \epsilon H_1 \\ H_0 &= -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m \omega_0^2 r^2 \\ \epsilon H_1 &= \frac{1}{3} m \omega_0^2 \epsilon (x^2 + y^2 - 2z^2) = -\frac{1}{3} \sqrt{\frac{16\pi}{5}} m \omega_0^2 \epsilon r^2 Y_{20}, \end{aligned} \quad (4.29)$$

where in the last step we have used the definition of the spherical harmonics that has been often employed in this Chapter. If the deformation is small, and ϵ is a small parameter, the effect of H_1 can be treated within first-order perturbation theory. The levels that are degenerate under the spherical assumption undergo a splitting that can be calculated in the following way:

$$\begin{aligned} \Delta \varepsilon_{nljm} &\equiv \langle nljm | \epsilon H_1 | nljm \rangle \\ &= -\frac{1}{3} \sqrt{\frac{16\pi}{5}} \epsilon m \omega_0^2 \int dr r^2 u_{nlj}^2(r) \langle ljm | Y_{20} | ljm \rangle \\ &= \frac{1}{3} \frac{\epsilon}{2} m \omega_0^2 \int dr r^2 u_{nlj}^2(r) \frac{3m^2 - j(j+1)}{j(j+1)}, \end{aligned} \quad (4.30)$$

where in the last step we have used some standard formula to calculate the expectation value of $Y_{20}(\hat{r})$ over a wave function of the type (2.5). The latter expression (4.30) predicts that states with $\pm m$ remain degenerate, as it can be expected in the case of the symmetry at hand where the nucleus is invariant when z' is reversed; moreover, at least for large j , the states with the maximum projection $m = \pm j$ are shifted up while the states with minimum projection $m = \pm 1/2$ are shifted down. This is understandable in semiclassical terms: the wave function should be concentrated in the plane perpendicular to the angular momentum and therefore in the case of maximum (minimum) projection over z' it is mainly in the plane $x'y'$ (along the z' axis) which is the region where the potential shrinks (becomes shallower).

There is another case that can be solved exactly, that is, the case of extremely large deformation in which one can assume a harmonic oscillator with cylindrical symmetry around the z' axis. Although the realistic cases are in between the two limits of the small and large deformations, the labels associated with the large deformation or cylindrical symmetry are often used throughout all values of ϵ . The exact quantum numbers are the projection of the total angular momentum on the z' axis (that has been called Ω in the above discussion) and the parity. Approximate quantum numbers are: the principal quantum number of the harmonic oscillator N , the number of nodes in the z' direction n_z , and the projection over z' of the spatial angular momentum Λ . The notation

$$\Omega^\pi [N, n_z, \Lambda]$$

is used when the levels corresponding to a potential having given deformation are calculated.

In reality, one solves the Hartree-Fock or BCS equations without assuming any symmetry. The nucleons find the arrangement that allows them to occupy states that are as lowest as possible in energy, that is, in order to minimise the total energy. In many instances, deformed solutions arise. They usually correspond to the cases in which both protons and neutrons have open shells.

Chapter 5

Nuclear Reactions

5.1 A general introduction

The domain of nuclear reactions is very broad, and the border between nuclear and hadronic reactions is not completely well defined. In the present Chapter, we will have in mind reactions where both the initial and final systems are either nucleons or nuclei, disregarding in this way the production of other particles. We will also leave aside electro-weak reactions, where electrons, muons or neutrinos interact with nuclei. In this respect, a typical nuclear reaction is

$$A + a \rightarrow B + b. \quad (5.1)$$

The presence of more than two particles in the final state cannot be excluded. In the initial state, reactions involving three or more particles have increasingly small cross sections. In any event, we shall keep the notation (5.1) for the sake of simplicity.

There are important conservation laws that dictate which reactions are possible, and which are the available final states for the particles that are involved: total energy, linear momentum, angular momentum, total electric charge, baryon number are conserved quantities, as well as parity since we neglect the weak interaction contribution. Energy conservation allows defining the reaction Q -value, that is,

$$Q \equiv (M_A + M_a - M_B - M_b) c^2 = \sum_f T_f - \sum_i T_i, \quad (5.2)$$

where M labels the mass of the particles, i and f label instead the particles in the initial and final states, respectively, with their kinetic energies T_i and T_f .

While the Q -value is of course a scalar quantity that does not depend on the chosen reference frame, the partition of kinetic energies among particles does. Although we will not detail here the formulas for the reaction kinematics, which can be found in many textbooks, we provide a few important comments on the laboratory and center-of-mass systems. As in previous Chapters, we stick to non-relativistic kinematics.

The laboratory (LAB) system is clearly defined. Usually, in it there is a target particle at rest¹ and this will be the particle A in the notation of Eq. (5.1). This is not the case in the center-of-mass (COM) frame. A simple calculation, carried out in the Appendix, leads to

$$\sum_i T_i^{(\text{COM})} = \frac{A}{A+a} T^{(\text{LAB})}, \quad (5.3)$$

where the l.h.s. is the total kinetic energy available in the COM system whereas at the r.h.s. we have the kinetic energy in the LAB system of the incident particle a, multiplied by a factor that includes the mass numbers of projectile and targets. In a reaction of a light particle, like a nucleon, against a heavy A, the two energies can be mixed up because the factor $\frac{A}{A+a}$ is close to 1: in fact, the COM essentially coincides with the heavy particle and remains idle, while this is not the case in a symmetric reaction where the energy in the COM is one half of the LAB energy. In what follows, we will study the properties of some reactions as a function of the energy, and is clear that such dynamics must be studied in the COM system and the COM energy will be referred to simply as *incident energy*. The associated velocity, or momentum, will be simply written as $\vec{v}$, or $\vec{p}$; we also shall use the name momentum, in this Chapter, for the wave vector $\vec{k}$ which is the momentum in units of $\hbar$. We shall keep implicit that in the COM system $T = \frac{1}{2}\mu v^2$ where μ is the reduced mass (cf. the Appendix).

The main observable associated with a reaction is its differential of total cross section². In keeping with the huge variety of possible reactions, the cross sections vary from negligibly small to large values³ of the order of $\approx$ barn. We remind that

$$1 \text{ b} = 10^{-24} \text{ cm}^2 = 100 \text{ fm}^2.$$

¹We leave aside the case of experiments performed with two colliding beams.

²The definitions are assumed to be known.

³On the nuclear scale.

We can classify the reactions according to their type or to the incident energy.

Elastic reactions are those in which neither the kind of particles nor the relative energy change between the initial and final state, namely we write $A + a \rightarrow A + a$. Inelastic reactions are those in which the kind of particles do not change, but some amount of energy is transferred from the relative kinetic energy to intrinsic excitations; in other words, one of the two particles (or both) can be excited. If we assume that A has been excited, we write $A + a \rightarrow A^* + a$. Transfer reactions are those in which B and b are different nuclei with respect to A and a , namely some nucleons have been transferred between projectile and target. We may have in mind one-neutron transfer, one proton-transfer, or multi-nucleon transfer.

At high energies, like 100 MeV/nucleon, nuclear reactions can be assumed to take place through one single step. The projectile is usually not much deviated from its trajectory, in a semiclassical picture, so that the angular distributions are forward-peaked and have specific shapes that are well reproduced by theory if we calculate them through first-order perturbation theory, namely by assuming one single interaction between projectile and target, as we shall discuss below. The fact that only one single interaction takes place can be roughly justified through an estimate of the interaction time, that is,

$$\begin{aligned} \Delta t &= \frac{R}{v} = \hbar \frac{R}{\hbar c} \sqrt{\frac{\mu c^2}{2T}} \approx 7 \cdot 10^{-22} \text{ MeV} \cdot \text{s} \frac{5 \text{ fm}}{200 \text{ MeV} \cdot \text{fm}} \sqrt{\frac{10^3}{2T}} \\ &\approx 10^{-23} \text{ s} \sqrt{\frac{10^3}{2T}}, \end{aligned} \quad (5.4)$$

where T is assumed in MeV. For T of the order of hundreds of MeV, this interaction time is of the order of the typical time between two nuclear collisions that has been estimated in Eq. (2.30). These reactions are called, therefore, *direct* reactions.

At the other extreme, reactions at 1 MeV/nucleon or so are characterised by interaction times that are one order of magnitude longer, or by a number of collisions that are one order of magnitude larger. Not surprisingly, as will be discussed in some detail below, in these reactions the large number of collisions produce an equilibrated, or thermalized, projectile-target compound system that is called *compound nucleus* (CN). The CN lasts for enough long time so to loose memory of the way it has been formed, and decays to a final state independently of the initial state. Only the energy deposited constitutes a relevant parameter. Cross sections are isotropic.

These two kinds of reactions will be discussed, while we leave aside the intermediate case of partial equilibration, the so called *multi-step* or *pre-equilibrium* reactions in keeping with their higher complexity. After this phenomenological introduction, we discuss some formal theory of the scattering process and a general formula for the associated cross section, that will be prone to several approximations.

5.2 Formal theory of nuclear reactions

We consider the two nuclei A and a, and we define in the following way the degrees of freedom and the associated Hamiltonian. $\vec{R}$ is the relative coordinate (i.e., the distance between the two center-of-mass), so that the relative kinetic energy can be written as $T = -\frac{\hbar^2}{2\mu}\nabla_{\vec{R}}^2$, where μ is the reduced mass and $\vec{\nabla}_{\vec{R}}$ is the gradient with respect to the coordinate $\vec{R}$. The coordinates of the nucleons belonging to A (a) are denoted by $\vec{r}_A$ ($\vec{r}_a$); accordingly, the intrinsic Hamiltonians will be written $H_A(\{\vec{r}_A\})$ and $H_a(\{\vec{r}_a\})$, and the dependence on the set of intrinsic coordinates will be dropped for simplicity. Of course, A will contain the kinetic energies of the nucleons of A in the system which is fixed in its center of mass, and the interaction between all these nucleons; the same will hold in the case of the nucleus a. Eventually, the remaining part of the Hamiltonian will be the interaction V_{Aa} between a nucleon belonging to A and a nucleon belonging to a. The terms of this potential will depend clearly on $\vec{R} + \vec{r}_a - \vec{r}_A$. This, as well as the rest of the previous discussion, should appear clearly from Fig. 5.1. It should be noted that the situation will be to some extent different in the case of transfer reactions; this case is not discussed here but one can consult specialized papers like T. Tamura, Phys. Rep. 14, 59 (1974), and in particular Fig. 1.

The proper Hamiltonian will then be

$$H = H_A + H_a + T + V_{Aa} = H_0 + V, \quad (5.5)$$

where, in the second step, we make it clear that we assume that H_A and H_a are known, together with their eigenvalues and eigenvectors, so that the problem in which $V \equiv V_{Aa}$ vanishes has a known solution. We would like to treat V as a perturbation. We start by calling Φ the eigenfunctions of H_0 . These wave functions are products of the intrinsic wave functions ϕ_A and ϕ_a of H_A and H_a , respectively, and of the eigenfunction of T which is a plane

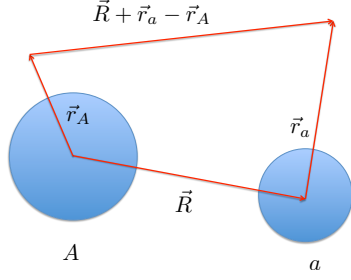


Figure 5.1: Appropriate coordinates for the collision between nuclei A and a.

wave. Thus, they read

$$\Phi_{\vec{k}}^{\pm}(\vec{R}) = e^{\pm i\vec{k} \cdot \vec{R}} \phi_A \phi_a, \quad (5.6)$$

where the + and - sign refers, respectively, to outgoing and ingoing boundary conditions (from now on, we will not carry on the $\pm$ notation as we will essentially have in mind the case of outgoing boundary conditions only). The equation

$$(E - H_0) \Phi_{\vec{k}} = 0 \quad (5.7)$$

holds, with

$$E = \frac{\hbar^2 k^2}{2\mu} + E_A + E_a. \quad (5.8)$$

The complete Schrödinger equation is

$$(E - H_0) \Psi_{\vec{k}} = V \Psi_{\vec{k}}, \quad (5.9)$$

namely we label as Ψ the exact wave function. $\Psi_{\vec{k}}$ will be written as the product of ϕ_A , ϕ_a and the exact wave function for the relative motion. Both Φ and Ψ depend both on the quantum number associated with the momentum, $\vec{k}$, and on two quantum numbers associated with the specific eigenstates of A and a (say, n_A and n_a). However, in what follows we keep these latter two quantum numbers as implicit and write only the momentum explicitly.

Along this line, $\Psi_{\vec{k}}$ can be expanded in terms of the set $\Phi_{\vec{k}'}$ in the following way:

$$\Psi_{\vec{k}}(\vec{R}) = \int d^3k' a(\vec{k}') \Phi_{\vec{k}'}(\vec{R}), \quad (5.10)$$

where a labels the (unknown) coefficients. We notice that the set Φ correspond of course to a continuum spectrum, and obey the completeness and closure relations in the following, generalised sense

$$\begin{aligned} \int d^3 R \Phi_{\vec{k}}^*(\vec{R}) \Phi_{\vec{k}'}(\vec{R}) &= (2\pi)^3 \delta(\vec{k} - \vec{k}'), \\ \int d^3 k \Phi_{\vec{k}}^*(\vec{R}) \Phi_{\vec{k}'}(\vec{R}') &= (2\pi)^3 \delta(\vec{R} - \vec{R}'). \end{aligned} \quad (5.11)$$

We can replace the expansion (5.10) into the l.h.s. of the Schrödinger equation (5.9), and use the fact that Φ is eigenfunction of H_0 , to get

$$\int d^3 k' (E - E') a(\vec{k}') \Phi_{\vec{k}'} = V \Psi_{\vec{k}}. \quad (5.12)$$

We now multiply at left both sides by $\Phi_{\vec{k}''}^*(\vec{R})$ and integrate over $\vec{R}$, to obtain, using the first of the equations (5.11),

$$(2\pi)^3 a(\vec{k}')(E - E') = \int d^3 R \Phi_{\vec{k}'}^*(\vec{R}) V(\vec{R}) \Psi_{\vec{k}}(\vec{R}). \quad (5.13)$$

We now use this equation to derive an expression for the coefficient a and we replace back this expression in the expansion (5.10), to get

$$\begin{aligned} \Psi_{\vec{k}}(\vec{R}) &= \int d^3 k' a(\vec{k}') \Phi_{\vec{k}'}(\vec{R}) = \int d^3 k' \int d^3 R' \frac{1}{(2\pi)^3} \frac{\Phi_{\vec{k}'}(\vec{R}) \Phi_{\vec{k}'}^*(\vec{R}')}{E - E'} V(\vec{R}') \Psi_{\vec{k}}(\vec{R}') \\ &= \int d^3 R' G_0(\vec{R}, \vec{R}'; E) V(\vec{R}') \Psi_{\vec{k}}(\vec{R}'). \end{aligned} \quad (5.14)$$

In the last step, we have defined the Green's function G_0 as

$$G_0(\vec{R}, \vec{R}'; E) = \frac{1}{(2\pi)^3} \int d^3 k' \frac{\Phi_{\vec{k}'}(\vec{R}) \Phi_{\vec{k}'}^*(\vec{R}')}{E - E'}. \quad (5.15)$$

Since the wave functions Φ are known, and E as well, the Green's function can be explicitly calculated. The result reads⁴

$$G_0(\vec{R}, \vec{R}'; E) = -\frac{\mu}{2\pi\hbar^2} \frac{e^{ik|\vec{R}-\vec{R}'|}}{|\vec{R} - \vec{R}'|}. \quad (5.16)$$

⁴It is an elementary result that can be derived by using the residue theorem, yet the derivation is available under request.

Eq. (5.14) shows one particular solution of the Schrödinger equation (5.9). The solution can be written as the general solution of the homogeneous equation, that is Φ , plus this particular solution:

$$\Psi_{\vec{k}} = \Phi_{\vec{k}} + \int d^3 R' G_0(\vec{R}, \vec{R}'; E) V(\vec{R}') \Psi_{\vec{k}}(\vec{R}'), \quad (5.17)$$

or, more schematically,

$$\Psi = \Phi + \int G_0 V \Psi. \quad (5.18)$$

This equation, (5.17) or (5.18), is called Lippmann-Schwinger equation. It is an integral equation which is equivalent to the Schrödinger equation we have started from, but its advantage is that it is written in a form that allows solutions by iteration, or anyway increasingly accurate solutions as a series in V ,

$$\Psi = \Phi + \int G_0 V \Phi + \int \int G_0 V G_0 V \Phi + \dots \quad (5.19)$$

This property will be used in what follows, where we actually shall stop, most of the time, at the lowest (i.e. Born) approximation where V appears only once. At the end of our derivation, we can re-introduce the internal coordinates of the two nuclei, $\vec{r}_1 \dots \vec{r}_a$ and $\vec{r}_1 \dots \vec{r}_A$ (cf. again Fig. 5.1). We also re-introduce the exact wave functions of the two nuclei ϕ_A and ϕ_a , either in their initial or final states. We do it while showing the relationship between the somehow formal Lippmann-Schwinger equation and the real experimental observable, that is, the cross section.

In the asymptotic region, $R \gg R'$, we can make the following approximations:

$$\begin{aligned} |\vec{R} - \vec{R}'|^{-1} &\approx \frac{1}{R}, \\ k|\vec{R} - \vec{R}'| &= k \left(R^2 + R'^2 - 2\vec{R} \cdot \vec{R}' \right)^{1/2} \approx kR \left(1 - \frac{\vec{R} \cdot \vec{R}'}{R^2} \right) = kR - \vec{k}' \cdot \vec{R}', \end{aligned} \quad (5.20)$$

where in the last step we have defined $\vec{k}'$ as the projection of $\vec{k}$ along the direction of $\vec{R}$, namely $\vec{k}' R \equiv k\vec{R}$. Such direction defines the scattering angle

θ , having assumed the initial momentum $\vec{k}$ along the z -axis. Therefore, we can write for $R \rightarrow \infty$

$$\begin{aligned} \Psi_{\vec{k}} \rightarrow e^{ikz} \phi_A \phi_a - \frac{\mu}{2\pi\hbar^2} \frac{e^{ikR}}{R} \int d^3R' e^{-i\vec{k}' \cdot \vec{R}'} \int d^3r_1 \dots d^3r_a d^3r_1 \dots d^3r_A \\ \phi_f^{(a)*}(\vec{r}_1 \dots \vec{r}_a) \phi_f^{(A)*}(\vec{r}_1 \dots \vec{r}_A) V(\vec{R}', \{\vec{r}\}) \phi_i^{(A)}(\vec{r}_1 \dots \vec{r}_A) \phi_i^{(a)}(\vec{r}_1 \dots \vec{r}_a) \Psi_{\vec{k}}(\vec{R}'). \end{aligned} \quad (5.21)$$

This equation allows one to identify an exact expression for the scattering amplitude, that is,

$$\begin{aligned} f(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d^3R' e^{-i\vec{k}' \cdot \vec{R}'} \int d^3r_1 \dots d^3r_a d^3r_1 \dots d^3r_A \\ \phi_f^{(a)*}(\vec{r}_1 \dots \vec{r}_a) \phi_f^{(A)*}(\vec{r}_1 \dots \vec{r}_A) V(\vec{R}', \{\vec{r}\}) \phi_i^{(A)}(\vec{r}_1 \dots \vec{r}_A) \phi_i^{(a)}(\vec{r}_1 \dots \vec{r}_a) \Psi_{\vec{k}}(\vec{R}'). \end{aligned} \quad (5.22)$$

It is usual in the literature to leave the explicit integration on the relative coordinate $\vec{R}'$ and to use bra and ket notation, $\langle$ and $\rangle$, for the integration on the internal coordinates. We shall also use the short-hand notations $\langle f|$ for $\phi_f^{(a)*} \phi_f^{(A)*}$, $|i\rangle$ for $\phi_i^{(A)} \phi_i^{(a)}$. Thus,

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d^3R' e^{-i\vec{k}' \cdot \vec{R}'} \langle f|V|i\rangle \Psi_{\vec{k}}(\vec{R}'). \quad (5.23)$$

It has to be stressed that this expression neglects the fact that the colliding particles may have spin. Aside from this, it is so far exact. In the case of elastic scattering i and f coincide with the ground-state of projectile and target, that will be written as $|0\rangle$. In such a case, the elastic cross section reads

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \quad (5.24)$$

We now discuss practical approximations and results for such elastic case.

5.3 Elastic scattering

In the case of elastic scattering, the transferred momentum $\vec{q}$ is defined by

$$\vec{q} \equiv \vec{k} - \vec{k}', \quad (5.25)$$

where $\vec{k}$ and $\vec{k}'$ are the initial and final momentum associated with the relative motion, as discussed in the previous Section, and they are equal in modulus. We can also write

$$\begin{aligned} q^2 &= k^2 + k'^2 - 2\vec{k} \cdot \vec{k}' \cos\theta = 2k^2 (1 - \cos\theta) \\ &= 4k^2 \sin^2 \frac{\theta}{2}, \\ q &= 2k \sin \frac{\theta}{2}. \end{aligned} \quad (5.26)$$

Thus, we have written for the elastic case the explicit form in which the angle θ appears in the scattering amplitude (cf. also Fig. 5.2).

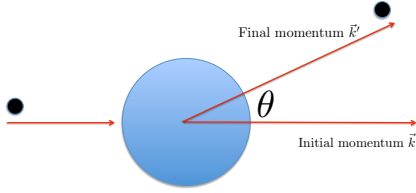


Figure 5.2: Initial and final momenta $\vec{k}$ and $\vec{k}'$. $\vec{q}$ is defined as $\vec{k} - \vec{k}'$ (see the text). θ is the scattering angle.

In the Born approximation the interaction V is allowed to act only once. The plane-wave Born approximation (PWBA) is the simplest possible approximation for elastic scattering: in it, starting from the exact expression for the scattering amplitude (5.23), the unknown wave function $\Psi_{\vec{k}}$ is replaced with the corresponding plane wave, that is,

$$\begin{aligned} f_{\text{PWBA}}(\theta) &= -\frac{\mu}{2\pi\hbar^2} \int d^3 R' e^{-i\vec{k}' \cdot \vec{R}'} \langle 0|V|0 \rangle e^{i\vec{k} \cdot \vec{R}'} \\ &= -\frac{\mu}{2\pi\hbar^2} \int d^3 R' e^{i\vec{q} \cdot \vec{R}'} \langle 0|V|0 \rangle. \end{aligned} \quad (5.27)$$

In the independent particle approximation,

$$\langle 0|V|0 \rangle = \sum_{Aa} \langle 0|V(\vec{R} + \vec{r}_a - \vec{r}_A)|0 \rangle = \int d^3 r_A d^3 r_a \rho_A(\vec{r}_A) V \rho_a(\vec{r}_a). \quad (5.28)$$

We can also mention the special case in which the projectile a is a point particle (a neutron or a proton⁵). Then,

$$\langle 0|V|0\rangle = \sum_A \langle 0|V(\vec{R} - \vec{r}_A)|0\rangle = \int d^3r_A \rho_A(\vec{r}_A)V \equiv U(\vec{R}). \quad (5.29)$$

As the latter equivalence hints, this is nothing but the mean field potential (at point $\vec{R}$). The scattering amplitude,

$$f_{\text{PWBA}}(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d^3R' e^{i\vec{q}\cdot\vec{R}'} U(\vec{R}'), \quad (5.30)$$

becomes simply the Fourier transform of the potential.

If we consider a neutron impinging on a nucleus, the potential could be represented with a Woods-Saxon shape or, more crudely, by a potential well. If the well has a radius a and a depth V_0 , the scattering amplitude becomes quite simply to evaluate. We introduce the development of the plane wave into spherical waves that will be useful many times later on

$$e^{i\vec{q}\cdot\vec{R}} = 4\pi \sum_{\lambda\mu} i^\lambda j_\lambda(qR) Y_{\lambda\mu}^*(\hat{q}) Y_{\lambda\mu}(\hat{R}), \quad (5.31)$$

where j_λ is the spherical Bessel function and the notation $\hat{q}$ ($\hat{R}$) corresponds to the polar angles of $\vec{q}$ ($\vec{R}$). Thus,

$$\begin{aligned} f_{\text{PWBA}}(\theta) &= -\frac{\mu V_0}{2\pi\hbar^2} \int_0^a dR R^2 d\hat{R} e^{i\vec{q}\cdot\vec{R}} = -\frac{2\mu V_0}{\hbar^2} \int_0^a dR R^2 j_0(qR) \\ &= -\frac{2\mu V_0}{\hbar^2 q^3} \int_0^{qa} dx x^2 j_0(x) = -\frac{2\mu V_0 a^2}{\hbar^2 q} j_1(qa), \end{aligned} \quad (5.32)$$

where we have used the expansion (5.31) in the first step, and the integral $\int dx x^2 j_0(x) = x^2 j_1(x)$ ⁶ in the last step. We can use Eq. (5.26) and, if we restrict to small angles, approximate

$$q = 2k \sin \frac{\theta}{2} \approx k\theta.$$

⁵In this latter case, we neglect the Coulomb potential which is indeed weak, and negligible with respect to the nuclear potential.

⁶Cf. e.g. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products. Seventh Edition* (Academic Press, 2007) Sec. 5.52.

In this way, the elastic cross section turns out to be

$$\frac{d\sigma}{d\Omega} = \frac{4\mu^2 V_0^2 a^4}{\hbar^4 k^2 \theta^2} |j_1(ka\theta)|^2. \quad (5.33)$$

The separation between two zeros of the Bessel function turns out to be $\delta\theta \approx \pi/ka$. The measured elastic cross section for neutrons on ^{208}Pb is displayed in Fig. 5.3. The value of k associated with neutrons impinging at 14.5 MeV is $\approx 0.83 \text{ fm}^{-1}$. If we set $a = 9 \text{ fm}$ we expect, therefore, a first zero around 24 degrees in the center-of-mass. This is well obeyed by the measured cross section. The formula we have discussed is actually analogous to what is known from basic diffraction theory. The radius of ^{208}Pb calculated by means of Eq. (2.1) is about 7 fm and it is not unreasonable to add the dimension of the neutron and the range of the interaction, about 1 fm each.

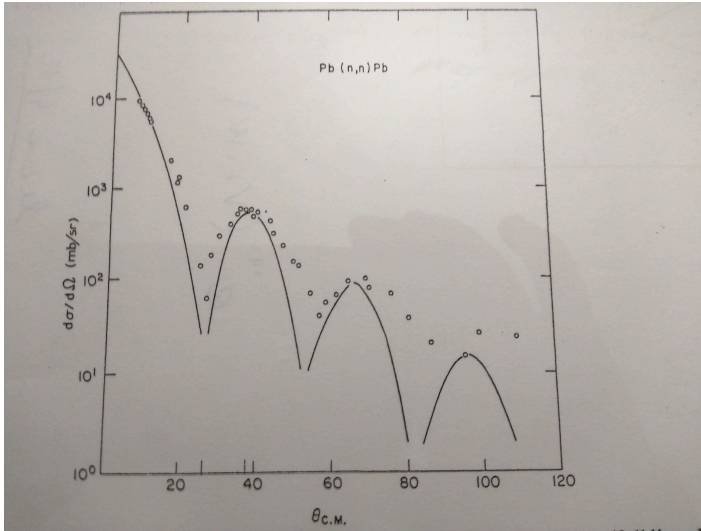


Figure 5.3: Elastic cross section for neutrons on Pb. The incident energy is about 14.5 MeV. The points correspond to measured values, whereas the line is a theoretical calculation.

The above picture makes sense when the so-called *absorption* is weak. It corresponds actually to treating the nucleus as a black sphere. There are many instances in which this is not true and PWBA does a quite poor job. The physical picture that corresponds to absorption and that is missing so far, is that only part of the incident neutrons undergo elastic scattering, that is, they remain in the *elastic channel*. The rest of the incident flux is

lost because neutrons perform inelastic or transfer reactions and should be found in other *reaction channels*. Usually this is dealt with by means of the introduction of an *optical potential*. Optical potentials replace the mean-field U and have the form

$$U_{\text{opt}} = V(\vec{R}) + iW(\vec{R}). \quad (5.34)$$

The imaginary potential corresponds to an absorption of the particles and is related to the concept of mean free path. We can easily understand that if we consider for a while a uniform medium like nuclear matter, where the wave functions can be assumed to be plane waves. The Schrödinger equation for a neutron in uniform matter reads

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V + iW\right)e^{ikz} = Ee^{ikz}, \quad (5.35)$$

where a momentum along the direction of z has been assumed for the sake of simplicity. Therefore,

$$\begin{aligned} \frac{\hbar^2 k^2}{2\mu} + V + iW &= E \\ k^2 &= \frac{2\mu}{\hbar^2} (E - V - iW). \end{aligned} \quad (5.36)$$

The quantity k becomes a complex number and we can extract it in the hypothesis (to be verified *a posteriori*) that in absolute value W is significantly smaller than $E - V$:

$$k = \sqrt{\frac{2\mu(E - V)}{\hbar^2}} \left(1 - i\frac{W}{E - V}\right)^{1/2} \approx \sqrt{\frac{2\mu(E - V)}{\hbar^2}} \left(1 - \frac{i}{2} \frac{W}{E - V}\right). \quad (5.37)$$

The momentum assumes an imaginary part, that is, the wave function becomes

$$e^{i(k_R + ik_I)z} = e^{ik_R z} e^{-k_I z}, \quad (5.38)$$

and its square modulus assumes a factor $e^{-2k_I z}$ that implicitly defines a *mean free path* l through the identification $e^{-2k_I z} \equiv e^{-z/l}$. Of course we expect W to be negative so that k_I is positive. We can write

$$l = \frac{1}{2k_I} = \sqrt{\frac{\hbar^2}{2\mu}} \frac{\sqrt{E - V}}{|W|}. \quad (5.39)$$

In Fig. 5.4 values for the real and imaginary parts of the optical potential are provided. In finite nuclei, such potentials may have a Woods-Saxon shape, although sometimes an imaginary part given by the derivative of a Woods-Saxon is preferred (the corresponding function is peaked on the surface and almost zero elsewhere, that is, it simulates a surface absorption).

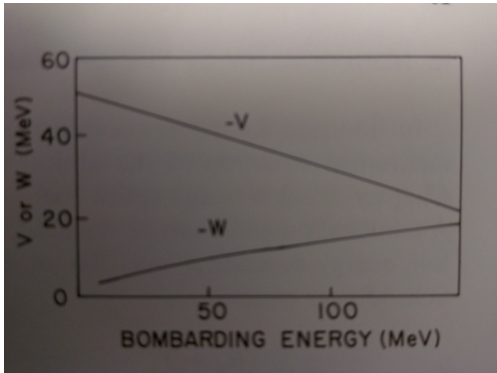


Figure 5.4: Schematic behaviour of the real and imaginary parts of the optical potential (5.34). Taken from N.K. Glendenning.

By using Eq. (5.39) and the values of Fig. 5.4, for neutrons of 50 MeV we find a mean free path around 4.5 fm. This is somewhat smaller than the value at lower energies that tends to be of the order of the radius of a heavy nucleus or larger, as we have discussed above.

We end this Section by reminding that we have not considered the effect of the Coulomb interaction. The exact scattering amplitude contains, in addition to the term associated with the nuclear interaction, the one associated with the Coulomb interaction. While the Coulomb part can be neglected for light particles, this is not the case for heavy ions. The interference between nuclear and Coulomb amplitudes gives rise to important effects whose

discussion is outside our present scope.

The calculations of elastic cross sections through optical potentials can be performed only numerically. It is instructive to discuss, in the next Section, a very simple approach for this aim.

5.4 Eikonal approximation

This approximation is, strictly speaking, valid at high energy. However, its advantage is that it provides a very simple expression for the elastic cross section, which is amenable to a quite straightforward numerical treatment. The name comes from optics, where it corresponds to the limit in which physical optics coincides with geometrical optics since the scale on which the refraction index n varies is large compared to the wavelength λ . In nuclear physics, this wavelength is the De Broglie one associated with the relative motion: if λ is small enough, or k is large enough, the approximation is valid. While for neutrons or protons it has been shown that one needs to go to quite large energies of the order of 500 MeV or 1 GeV, the same approximation can be safely used at much lower energies in the case of heavy ions; in fact k scales with $\sqrt{A}$.

In the eikonal approximation one assumes trajectories that are straight lines. The wave function of the initial state, assumed to have momentum along the z -axis, has the form e^{ikz} times a minimal distortion given by a scalar function. Cylindrical coordinates will be used, that is, z and the two-dimensional vector $\vec{b}$ on the orthogonal plane (it is written in this form because it coincides with the impact parameter). The rotation angle around z will not play any role. The function $\Psi_{\vec{k}}$ is therefore written as⁷

$$\Psi_{\vec{k}}(\vec{R}) = e^{ikz} \varphi(z, b). \quad (5.40)$$

We replace this wave function in the Schrödinger equation for the relative motion,

$$-\frac{\hbar^2}{2\mu} \nabla^2 \Psi_{\vec{k}} + V \Psi_{\vec{k}} = E \Psi_{\vec{k}}, \quad (5.41)$$

and we make the assumption that the term containing the Laplacian of φ can be neglected, namely that

$$|\nabla^2 \varphi| \ll k |\vec{\nabla} \varphi|.$$

⁷As above, we keep intrinsic coordinates and intrinsic wave functions implicit until needed.

This equation represents, in mathematical terms, the fact that the distortion is minimal, and is eventually valid for large enough values of k . It is easy to verify that the Schrödinger equation simplifies to

$$\frac{\partial \varphi}{\partial z} = -\frac{i}{\hbar v} V \varphi, \quad (5.42)$$

with $v = \hbar k / \mu$. The solution reads

$$\varphi(b, z) = \exp \left[-\frac{i}{\hbar v} \int_{-\infty}^z dz' V(b, z') \right] = e^{i\chi(b, z)}. \quad (5.43)$$

This is a very simple equation, in which the second step is just a definition of the quantity $\chi(b, z)$. Re-inserting at this point intrinsic coordinates and wave functions the proper definition of $\chi(b, z)$ becomes

$$\chi(b, z) = -\frac{1}{\hbar v} \int_{-\infty}^z dz' \langle f | V(b, z') | i \rangle. \quad (5.44)$$

We now show how to deduce the elastic cross section. In this case, the quantity that enters the above equation for $\chi(b, z)$ is $\langle 0 | V | 0 \rangle$, and this can be written as U as we did in the previous discussion.

$$\begin{aligned} f_{\text{eiko}}(\theta) &= -\frac{\mu}{2\pi\hbar^2} \int d^3 R' e^{-i\vec{k}' \cdot \vec{R}'} U e^{ikz} e^{i\chi(b, z)} \\ &= -\frac{\mu}{2\pi\hbar^2} \int d^2 b e^{i(\vec{k} - \vec{k}') \cdot \vec{b}} \int dz e^{i(k_z - k'_z)z} U e^{i\chi(b, z)}. \end{aligned} \quad (5.45)$$

We shall use the approximation that the momentum variation is small enough so that it is perpendicular to z . Then, $\vec{k} - \vec{k}'$ has only component parallel to $\vec{b}$ and the last integral of the equation simplifies to

$$\begin{aligned} \int_{-\infty}^{+\infty} dz U e^{-\frac{i}{\hbar v} \int_{-\infty}^z dz' U} &= i\hbar v \left(e^{-\frac{i}{\hbar v} \int_{-\infty}^{+\infty} dz' U} - 1 \right) \\ &= i\hbar v (e^{i\chi(b)} - 1), \end{aligned} \quad (5.46)$$

where the last step is a definition of $\chi(b) = \chi(b, z = +\infty)$. With this simplification, and the usual definition of the momentum transfer already given,

$$f_{\text{eiko}}(\theta) = -\frac{ik}{2\pi} \int d^2 b e^{i\vec{q} \cdot \vec{b}} (e^{i\chi(b)} - 1). \quad (5.47)$$

Finally, if ϕ is the angle around z , we express the integral in d^2b as $db\, b\, d\phi$ and we use the definition of the Bessel function J_0 in terms of

$$\int_0^{2\pi} d\phi\, e^{iqb\cos\phi} = 2\pi J_0(qb).$$

We arrive at the final equation

$$f_{\text{eiko}}(\theta) = -ik \int db\, b\, J_0(qb) (e^{i\chi(b)} - 1). \quad (5.48)$$

The two equations (5.43) and (5.48) provide the full solution of the elastic cross section in the eikonal approximation, by means of just two simple one-dimensional integrals. Of course the elastic cross section is just $|f(\theta)|^2$.

5.5 Inelastic scattering

5.5.1 General considerations and kinematics

Inelastic scattering experiments are, as a rule, carried out with the aim of obtaining information on the excitation spectra of nuclei (vibrational states, rotational bands, or other kinds of excitations). The main problem is the reaction selectivity. If we produce a beam with given energy, and the particles interact with target nuclei, in the simplest possible experimental set-up only the energy lost by the projectile is measured, together with the scattering angle. First, it would be relevant to make sure that only one of the two nuclei, say the target⁸ is excited. To this aim, one can use as a projectile either protons, or α -particles that do not have excited states below 20 MeV, or other nuclei that allow distinguishing in some way the case of projectile excitation⁹.

Another issue is that, even if one is sure to deal only with target excitation, the identification of the quantum numbers of the excited states is not straightforward. At times, this can be done only with the guidance of theoretical calculations, or of systematics. In the case of spherical nuclei, J and

⁸We have in mind the usual configuration in which stable nuclei are studied. Of course, in the case of exotic nuclei that produced in beam, the situation is reversed and the roles of projectile and target in this and in the following sentences should be exchanged.

⁹In the case of ^{17}O , just to make one example, the low energy threshold for neutron emission allows studying the target excitations above that threshold without ambiguity.

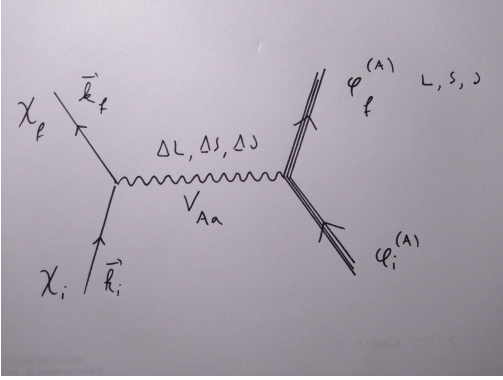


Figure 5.5: Representation in terms of a Feynman diagram of the projectile-target interaction in the case of inelastic scattering. The notation will become clear as the discussion in the main text will be developed. We are assuming that the initial state of the target, whose associated wave function is $\phi_i^{(A)}$, has $L = S = J = 0$ and that the final state, $\phi_f^{(A)}$, has finite values of those quantum numbers that coincide with the quantum numbers labelled by Δ , namely those which are transferred through the projectile-target interaction.

the parity π are good quantum numbers. To a good approximation, we can consider L and S , together with J , as good quantum numbers. If the nucleus is even-even, $J = L = S = 0$ in the ground-state, and we can identify J , L and S with the quantum numbers exchanged in the projectile-target interaction that can be labelled as ΔJ , ΔL and ΔS respectively (cf. Fig. 5.5). We may add isospin to this discussion although we shall not do it for the sake of simplicity. T and ΔT would be in this case the isospin of the target and the transfer via the projectile-target interaction. The main specific aspects that are at variance with the discussion above are that most of the nuclei have $T \neq 0$ in the ground state, and that the assumption that isospin is an approximate quantum number is only valid in light nuclei (isospin is badly broken by the Coulomb interaction as Z increases).

The selectivity in ΔS can be achieved in few cases. For instance, α -particles are widely used in inelastic scattering experiments because they lead exclusively to $S = 0$ states¹⁰. As for ΔL , in order to estimate it we can use semi-classical considerations. We shall use such considerations later on as well. The angular momentum of the relative motion will be written in

¹⁰ $T = 0$ is also guaranteed.

absolute value as kb where b is the impact parameter, so that

$$\Delta L = \Delta k \cdot b = qR, \quad (5.49)$$

where in the last step we have assumed a grazing collision and replaced the impact parameter with the target radius. We make another approximation in this Section. We assume that inelastic scattering is performed at sufficiently high energy so that the absolute value of $\vec{k}_f$, although smaller than $\vec{k}_i$, can be confused with it. Thus, using a common symbol k for the absolute value of those vectors,

$$\Delta L^2 = (k_i^2 + k_f^2 - 2\vec{k}_i \cdot \vec{k}_f \cos\theta)R^2 \approx 4k^2 \sin^2 \frac{\theta}{2} R^2. \quad (5.50)$$

For small angles, where $\sin\theta$ can be replaced by θ ,

$$\Delta L \approx kR\theta. \quad (5.51)$$

One can identify ΔL from the angle θ at which the angular distribution has a maximum for given excitation energy (the excitation energy coincides with the energy lost by the projectile).

As an example, we could consider α particles having energy $E = 172$ MeV. The value of k can be calculated from $\sqrt{2\mu E}/\hbar$ and turns out to be approximately 5.8 fm^{-1} . From Eq. (5.51), we learn that for e.g. Sn nuclei that have a radius of about 6 fm, the cross section should be peaked at 3.3 degrees for $\Delta L = 2$. In this way, it has been possible to identify the main peak appearing in Fig. 5.6 as the Giant Quadrupole Resonance as it shows up at 4 degrees.

5.5.2 General formulas for the cross section

We start again from the general expression for the scattering amplitude (5.23) that we have provided above:

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d^3R' e^{-i\vec{k}' \cdot \vec{R}'} \langle f|V|i \rangle \Psi_{\vec{k}}(\vec{R}').$$

Now i and f are typically the ground-state and some excited state, respectively. In the present case, the quantity $\langle f|V|i \rangle$ is often called form factor or transition amplitude T_{fi} . As above, it depends only on the coordinate $\vec{R}'$ after the integration on the internal coordinates has been performed.

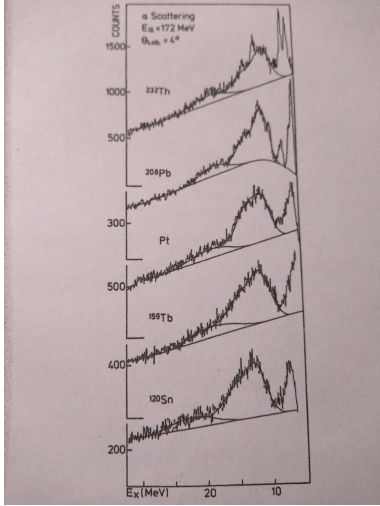


Figure 5.6: Energy spectra of inelastically scattered α -particles having a bombarding energy of 172 MeV and a scattering angle of 4 degrees. Taken from M. Harakeh and A. Van der Woude.

There exist, as in the case of elastic scattering, several possible approximations in keeping with the fact that the exact wave function Ψ is in general unknown. PWBA or eikonal approximations can be used. We will not repeat the discussion here because it follows the line of the one carried out in the elastic case. In particular, in the case of PWBA, the scattering amplitude will be proportional to the Fourier transform of the form factor.

The most widely used approximation is the Distorted Wave Born Approximation (DWBA). It relies on the numerical solution of the elastic scattering with an Hamiltonian

$$H = H_0 + V_{\text{opt}}, \quad (5.52)$$

where H_0 has been given in Eq. (5.5) (it includes the kinetic energy of the relative motion and the two intrinsic Hamiltonian) while V_{opt} is an effective interaction that is chosen so that the elastic scattering solution is known and reproduces experiment at best. In other words, this operator is chosen so that $\langle 0|V_{\text{opt}}|0\rangle$ corresponds to U_{opt} as in Eq. (5.34). The complete Hamiltonian will be

$$H = H_0 + V_{\text{opt}} + (V - V_{\text{opt}}), \quad (5.53)$$

and the treatment of Sec. 5.2 can be repeated as it is but with H_0 replaced

by $H_0 + V_{\text{opt}}$ in the role of the unperturbed Hamiltonian and V replaced by $V - V_{\text{opt}}$ in the role of the perturbation. The eigenfunctions of $H_0 + V_{\text{opt}}$, that replace the plane waves eigenfunctions of H_0 , are usually denoted by χ and called *distorted waves*. The quantity $V - V_{\text{opt}}$ is written as V for simplicity. If we then write the corresponding equation of (5.23) with the unknown wave function Ψ replaced by χ , and $V - V_{\text{opt}}$ rewritten simply as V , we obtain

$$f_{\text{DWBA}}(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d^3R' \chi_{\vec{k}'}^*(\vec{R}') \langle f|V|i\rangle \chi_{\vec{k}}(\vec{R}'). \quad (5.54)$$

This is the DWBA expression for the transition amplitude.

Within all possible approximations, we remind that the cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{k_f}{k_i} |f(\theta)|^2. \quad (5.55)$$

We continue with two specific examples. In the next subsection we treat the special case of Coulomb excitation, in which the external perturbation on the excited nucleus is exactly known. In the next subsection, we show how within PWBA the nuclear excitation can be treated in a simple way, so that the excitation cross section is proportional to the strength associated with simple operators.

5.5.3 Coulomb excitation

We assume in this subsection that we are able to single out scattering events in which the impact parameter is large and thus, only the long-range Coulomb force is active. We also assume that the projectile remains in its ground-state, and its action within the form factor is the same as that of a point charge Z_a . Under this hypothesis, we can write the form factor in terms of the degrees of freedom of the target only, that is,

$$\begin{aligned} T_{fi} &\equiv \langle f|V_{\text{Coul}}|i\rangle \\ &= \int d^3r_1 \dots d^3r_A \phi_f^{(A)*}(\vec{r}_1 \dots \vec{r}_A) V_{\text{Coul}} \phi_i^{(A)}(\vec{r}_1 \dots \vec{r}_A). \end{aligned} \quad (5.56)$$

We work again within the independent particle approximation. The description of nuclear excited states in terms of a superposition of single-particle excitations has already been carried out in Sec. 3.4. In this spirit, the operator V_{Coul} plays exactly the role of F , that has been introduced in Eq.

(3.4). If we call $|LM\rangle$ the excited state (using the quantum numbers that are appropriate in the case of a spherical nucleus), and we use (3.29) and the related equations, then

$$\begin{aligned} T_{fi} &= \langle LM | V_{\text{Coul}} | 0 \rangle = \sum_{\text{ph}} X_{\text{ph}} \langle ph^{-1} | V_{\text{Coul}} | 0 \rangle \\ &= \sum_{\text{ph}} \langle 0 | \delta\rho | ph^{-1} \rangle \langle ph^{-1} | V_{\text{Coul}} | 0 \rangle = \int d^3r V_{\text{Coul}}(\vec{R} - \vec{r}) \delta\rho_{LM}(\vec{r}), \end{aligned} \quad (5.57)$$

where the completeness relation has been used in the last step. The quantity $\delta\rho$ is usually called *transition density*. It must have the form

$$\delta\rho(\vec{r}) = \delta\rho_{LM}(\vec{r}) = \delta\rho_{LM}(r) Y_{LM}^*(\hat{r}), \quad (5.58)$$

because the density variation associated with the creation (annihilation) of a state with angular momentum LM must involve the function Y_{LM}^* (Y_{LM}).

It is useful, at this point, to expand the Coulomb interaction into multipoles,

$$\begin{aligned} V_{\text{Coul}}(\vec{R} - \vec{r}) &= \frac{e^2 Z_A Z_a}{|\vec{R} - \vec{r}|} \\ &= e^2 Z_A Z_a \sum_{\lambda\mu} \frac{4\pi}{2\lambda + 1} \frac{r_{<}^\lambda}{r_{>}^{\lambda+1}} Y_{\lambda\mu}^*(\hat{R}) Y_{\lambda\mu}(\hat{r}), \end{aligned} \quad (5.59)$$

where $r_{<}$ ($r_{>}$) are the smaller (larger) between r and R . In the case under study, clearly $r_{>} = R$ and $r_{<} = r$, so that the previous equation (5.57) becomes

$$\begin{aligned} T_{fi} &= e^2 Z_A Z_a \sum_{\lambda\mu} \frac{4\pi}{2\lambda + 1} \int d\hat{r} Y_{\lambda\mu}(\hat{r}) Y_{LM}^*(\hat{r}) \frac{1}{R^{\lambda+1}} Y_{\lambda\mu}^*(\hat{R}) \int dr r^{\lambda+2} \delta\rho_{LM}(r) \\ &= \frac{4\pi e^2 Z_A Z_a}{2L + 1} \frac{1}{R^{L+1}} Y_{LM}^*(\hat{R}) \int dr r^{L+2} \delta\rho_{LM}(r). \end{aligned} \quad (5.60)$$

If this form factor is inserted into a reliable expression for the scattering amplitude, it is clear that the only unknown is the transition density. In other words, due to the fact that the Coulomb interaction is known, the measurement of the cross section for Coulomb excitation allows a direct determination of the transition density of the excited state.

An example is that of the excitation of small-amplitude surface modes. A small variation of the density can be taken as proportional to its derivative, that is, one can assume

$$\delta\rho_{LM} \equiv \beta_L \frac{d\rho_0}{dr}, \quad (5.61)$$

where ρ_0 is the g.s. density and β_L is the only free parameter, that can be determined by means of a fit to experimental data.

5.5.4 Nuclear excitation within PWBA with a zero-range force

The use of the PWBA and of a projectile-target interaction which is proportional to a Dirac delta is not realistic but has a pedagogical value, as we will show at the end. The interaction V can be expanded in multipoles, in analogy with Eq. (5.59), as

$$V(\vec{R} - \vec{r}) = V_0 \delta(\vec{R} - \vec{r}) = \sum_{\lambda\mu} \frac{\delta(r - R)}{r^2} Y_{\lambda\mu}(\vec{r}) Y_{\lambda\mu}^*(\vec{R}). \quad (5.62)$$

The form factor can be obtained in the same way that has led to Eq. (5.60) and becomes

$$T_{fi} = V_0 \delta\rho_{LM}(R) Y_{LM}^*(\hat{R}). \quad (5.63)$$

We now calculate the scattering amplitude within PWBA, that is,

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d^3R' e^{i\vec{q}\cdot\vec{R}'} T_{fi}(\vec{R}'). \quad (5.64)$$

We employ, accordingly, the development (5.31) and obtain

$$f(\theta) = -\frac{2\mu}{\hbar^2} i^L Y_{LM}^*(\hat{q}) V_0 \int dR' R'^2 j_L(qR) \delta\rho_{LM}(R). \quad (5.65)$$

Often, the momentum transfer is such that over the distance scale in which the transition density is not negligible (typically, the scale associated with the nuclear surface) the condition $qR \ll 1$ holds. The spherical Bessel functions for small arguments behave as $j_L(qR) \approx (qR)^L$, and the scattering amplitude can be written

$$f(\theta) = -\frac{2\mu}{\hbar^2} i^L q^L V_0 Y_{LM}^*(\hat{q}) \int dR' R'^{L+2} \delta\rho_{LM}(R), \quad (5.66)$$

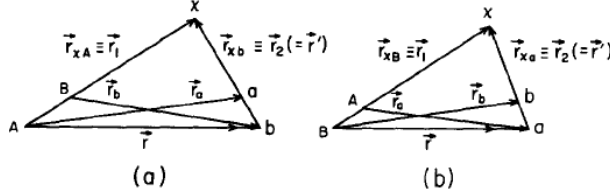


Figure 5.7: Choice of coordinates appropriate for (a) stripping and (b) pick-up reactions. Figure taken from T. Tamura.

so that the corresponding cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{k_f}{k_i} \frac{4\mu^2 q^{2L} V_0}{(2L+1)\hbar^4} \sum_M Y_{LM}^*(\hat{q}) Y_{LM}(\hat{q}) \left| \int dR' R'^2 \delta\rho_{LM}(R) \right|^2. \quad (5.67)$$

We have supposed that the cross section is averaged over all possible final M -substates since we are unable to distinguish them with unpolarised beams and targets. It turns out that the cross section is proportional, via known factors, to the square of the radial matrix element associated with the action of an operator like R'^L , like in the electromagnetic case. This decomposition in two factors, a kinematical one and the square of the action of a radial multipole operator, is strictly speaking only valid within this not fully realistic plane-wave approximation; however, it possesses a conceptual validity as a guideline even in more realistic cases.

5.6 Transfer reactions

The treatment of these reactions is considerably more complicated than that of elastic and inelastic ones. At variance with them, in fact, the two nuclei do not simply interact by exchanging virtual particles or photons, but exchange real particles between them. The nucleons are in different partitions in the initial Aa and final Bb channels. One could define a natural coordinate system in each of these two initial and final channels, and in this way identify the two relative coordinates $\vec{R}_a$ and $\vec{R}_b$ in such channels (see Fig. 5.7 for the sake of illustration). The projectile-target interaction can be defined either

in the initial or final channel (where the words ejectile-residual could be more appropriate), and one refers to the two choices as prior or post representations, respectively. The exact treatment of prior and post representation for the transition amplitude goes beyond the scope of these lectures. We shall, therefore, treat a simple one-particle transfer case, namely the (p,d) reaction, within very crude approximations that leave completely aside the aforementioned difficulties.

Before that, we discuss for the case of heavy-ions some kinematical conditions on the transfer reactions in order to understand the best incident energies at which transfer cross sections are not negligibly small. Also in this case, we shall have only the one particle-transfer case in mind. Multi-particle transfer reactions, although of interest, will not be treated.

5.6.1 Kinematical effects on transfer reactions

In the case of charged particles, the transfer reactions at energies below the Coulomb barrier suffer from the hindrance due to the tunnelling. For energies above the Coulomb barrier, instead, cross sections have a large value only if appropriate kinematical conditions are satisfied. As first discussed by D.M. Brink, we can neglect spin and assume that nucleons are treated classically. In such a case, the transfer happens preferably if the momentum of the transferred nucleon do not change much between initial and final state.

We define the geometry of the problem as in Fig. 5.8. The nucleus a is assumed to move along the y -axis in the lab system with velocity v . The nucleon, or cluster, is transferred from a to A and within this two nuclei is supposed to be in an orbital lying on the xy plane and having accordingly angular momentum along the z -direction. This angular momentum will be, respectively, $(\lambda_1, \mu_1 = \lambda_1)$ and $(\lambda_2, \mu_2 = \lambda_2)$ in a and A . We assume a grazing collision in which the nucleon is transferred when the two nuclei touch each other in a point on the y -axis and they find themselves at a distance R which is approximately equal to the sum of the two radii. In this situation, the initial and final momenta of the nucleons are

$$\begin{aligned} k_{i,y} &= \frac{mv}{\hbar} - \frac{\lambda_1}{R_1}, \\ k_{f,y} &= \frac{\lambda_2}{R_2}. \end{aligned} \tag{5.68}$$

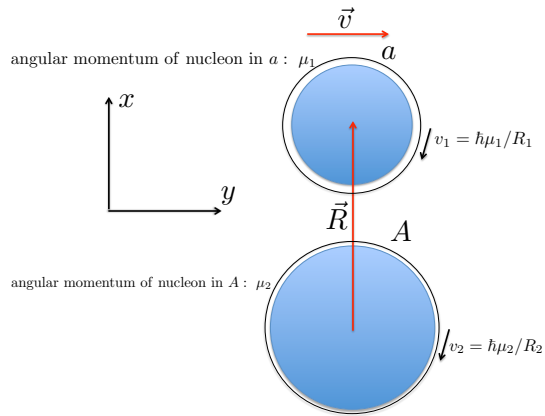
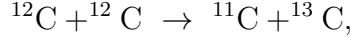


Figure 5.8: Classical description of the transfer of a nucleon, or a cluster, from a to A . In the figure we show the distance $\vec{R}$ between the two nuclei, the velocity $\vec{v}$ of the nucleus a , the classical orbits of the transferred nucleon in the two nuclei with the associated velocities as a function of the angular momentum, and the system of coordinates we wish to use. See the text for a more complete discussion.

The kinematical condition we would like to fulfill is, therefore,

$$\frac{\lambda_2}{R_2} + \frac{\lambda_1}{R_1} - \frac{mv}{\hbar} \approx 0. \quad (5.69)$$

If we apply it to the reaction



we should (a) use a value for R like 2.98 fm [from Eq. (2.1) by using 1.3 fm for r_0], (b) know the values of λ . In the g.s. to g.s. transition the neutron is transferred from $p_{3/2}$ to $p_{1/2}$. Application of Eq. (5.69) to this case gives $v/c \approx 0.14$ which corresponds to an incident energy of about 9.34 MeV/nucleon, which means around 112 MeV. A more realistic estimate could be obtained with a slightly larger value of R that takes into account the range of the nuclear interaction. However, these g.s. to g.s. transitions have been found to be quite strong in experiments at an incident energy of 114 MeV, so that our estimate is quite fair.

5.6.2 Simplified calculation of the PWBA transition amplitude

We consider the neutron pick-up reaction, that is,

$$A + p \rightarrow B + d. \quad (5.70)$$

Our assumption is that the nucleus A is well described by an unpaired neutron occupying an orbital with wave function $\phi_n(\vec{r}_n)$, with the $A - 1$ remaining nucleons forming a rather stiff core that acts as a spectator in the reaction. The set of coordinates of these nucleons will be simply labelled by $\vec{\xi} = \{\vec{r}_1 \dots \vec{r}_{A-1}\}$. The formulation of these hypotheses leads to

$$\Psi_A = \phi_n(\vec{r}_n) \Psi_{A-1}(\vec{\xi}) \quad (5.71)$$

(note the lack of antisymmetrization). The PWBA scattering amplitude can be written as

$$\begin{aligned} f_{\text{PWBA}}(\theta) &= -\frac{\mu}{2\pi\hbar^2} \int d^3R' \int d^3r_n d^3\xi e^{-i\vec{k}'_d \cdot \vec{r}_d} \phi_d^*(\vec{r}_p, \vec{r}_n) \Psi_B^*(\vec{\xi}) \\ &\times V(\vec{r}_p, \vec{r}_n) \phi_n(\vec{r}_n) \Psi_{A-1}(\vec{\xi}) e^{i\vec{k}_p \cdot \vec{r}_p}, \end{aligned} \quad (5.72)$$

where the labels p, d, n indicate obviously the proton, deuteron, and picked-up neutron, and the interaction has been identified with the p-n interaction. We make the so-called zero-range approximation. It is very crude and it consists in approximating the deuteron as a point-object. If ϕ_d is, approximately, proportional to a narrow function almost like $\delta(\vec{r}_p - \vec{r}_n)$, then we can set

$$\begin{aligned}\vec{r}_d &\approx \frac{1}{2}(\vec{r}_p + \vec{r}_n) \approx \vec{r}_n \\ \vec{R}' &\approx \vec{r}_d \approx \vec{r}_p \approx \vec{r}_n \\ \int d^3r_p V(\vec{r}_p, \vec{r}_n) \phi_d^*(\vec{r}_p, \vec{r}_n) &\approx D_0.\end{aligned}\tag{5.73}$$

The reason why we believe this approximation is quite crude is that we know that the deuteron is not a small object. However, with such approximations we can write easily the scattering amplitude avoiding the complications we have mentioned at the start of the Section. The scattering amplitude becomes

$$f_{\text{PWBA}}(\theta) = -\frac{\mu D_0}{2\pi\hbar^2} \int d^3r_n e^{i\vec{q}\cdot\vec{r}_n} \int d^3\xi \Psi_B^*(\vec{\xi}) \phi_n(\vec{r}_n) \Psi_{A-1}(\vec{\xi}).\tag{5.74}$$

The quantity

$$S = \int d^3\xi \Psi_B^*(\vec{\xi}) \Psi_{A-1}(\vec{\xi})\tag{5.75}$$

could be approximated as 1 if the core is really untouched by the pick-up of one nucleon. More realistically, we can assume that such component of the wave function, albeit a major one, is not the only one. In this case S is a number close, yet smaller than, one. It can be called *spectroscopic factor* associated with the state n . The reader can note, by going back to Sec. 2.3, that in the BCS language this number would correspond to the probability amplitude that the state n is occupied in the nucleus A , that is, we could call it v_n although this would be a model-dependent definition.

We now write the single-particle wave function of the state n by neglecting the spin degree of freedom as

$$\phi_n(\vec{r}_n) = \frac{u_{nl}(r)}{r} Y_{lm}(\hat{r}),\tag{5.76}$$

we insert this equation together with the previous equation (5.75) and the development (5.31) into Eq. (5.74), and we obtain

$$\begin{aligned} f_{\text{PWBA}}(\theta) &= -\frac{\mu D_0 S}{2\pi\hbar^2} \int d^3r_n \phi_n(\vec{r}_n) e^{i\vec{q}\cdot\vec{r}_n} \\ &= -\frac{2\mu D_0 S i^l}{\hbar^2} \int dr_n r_n u_{nl}(r) j_l(qr_n) \times Y_{lm}(\hat{q}). \end{aligned} \quad (5.77)$$

This equation has a quite transparent meaning, as it shows that the scattering amplitude is larger, in the PWBA, if the neutron wave function has a good overlap with the component of the plane wave that has the same angular momentum l , that is, with $j_l(qr_n)$. Needless to say, this is the quantum version of the semiclassical condition (5.69). More importantly, Eq. (5.77) provides an expression that can be calculated if one has a model for the wave function of the transferred neutron, like e.g. if one approximates $u_{nl}(r)$ with the solution of a Woods-Saxon potential of the type (2.3). The biggest limitation of Eq. (5.77) is the use of D_0 . This problem may be circumvented if one compares theory not with the absolute values of the cross sections but rather with relative values, namely ratios between cross sections associated with the transfer of nucleons from different nl states. In this case, D_0 can be simplified. In such situation, one usually attempts the comparison of experimental and theoretical cross sections in order to extract the spectroscopic factor S which remains the only unknown quantities. Even with models that are more realistic than that described in the current subsection, approximations are unavoidable and lead to errors on S that range between 10-15% and 30%. This is the present limit of our possibility to experimentally test the independent-particle description of nuclei.

We further approximate the latter equation (5.77) by assuming that the wave function is surface-peaked. In the limit in which u_n is proportional to $\delta(r_n - R)$ where R is here the nuclear radius, one arrives at

$$f_{\text{PWBA}}(\theta) \sim R j_l(qR), \quad (5.78)$$

where constant factors have been dropped for the sake of simplicity. This scattering amplitude can be again interpreted in diffractive terms. The associated cross section would be proportional to the square of a spherical Bessel function. This is evident in Fig. 5.10, if one compares the PWBA result that is displayed with the squares of the Bessel functions that are depicted in Fig. 5.9. Fig. 5.10 shows also experimental data, as well as the results from a

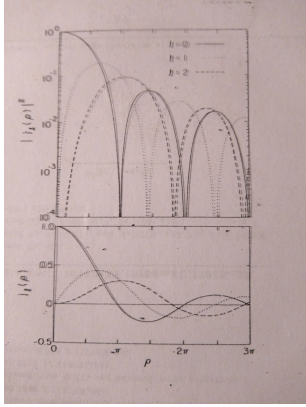


Figure 5.9: Spherical Bessel functions $j_l(qr)$ and their square. Note the similarity of the shape of $|j_l(qr)|^2$ and the PWBA cross section shown in Fig. 5.10 by means of a dashed line. This similarity reflects the result of Eq. (5.78).

more elaborate theory that takes into account all what we have neglected in the discussion of this subsection.

5.7 Compound nucleus reactions

At low energy, particles that participate to the reaction have quite a long time to interact with one another. As a consequence, one can imagine that a complete thermalization takes place, that is, the energy brought into the projectile plus target compound system by the relative motion is shared by all nucleons, like in the case of the equipartition that defines the concept of temperature. This picture, in which the system that has been formed has also, by definition, lost memory of the initial reaction, because only the total excitation energy has remained as a signature of the previous history, is the so called *compound nucleus* hypothesis. There are experimental evidences that this happens in many cases, so that the concept of nuclear temperature can be defined. Before coming to it in some detail, as we shall do in subsec. 5.7.3, we describe neutron reactions and charged particles reactions in subsecs. 5.7.1 and 5.7.2, respectively. There is a crucial difference between the two cases since these reactions take place at incident energies below the

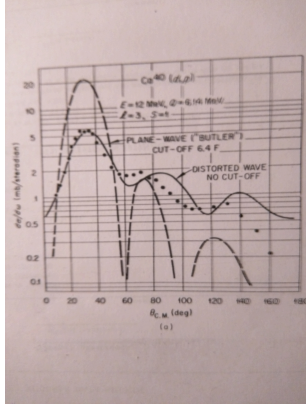


Figure 5.10: Calculated cross sections for the neutron pick-up on ^{40}Ca . PWBA and DWBA results are compared with the experimental data.

Coulomb barrier. What is common to the different cases we will see below, is that compound nucleus reactions are isotropic (memory is also lost regarding the incident projectile direction). Thus, we shall discuss total cross sections instead of angular distributions.

5.7.1 Neutron reactions

We shall discuss them by means of one example, and extrapolating from it their general features. The example of neutrons on ^{16}O is displayed in Fig. 5.11. The main feature of the cross section is the presence of narrow peaks, called *resonances*. We expect them to correspond to excited states in ^{17}O . The relation between the incident neutron energy T_n and the excitation energy E_x in this nucleus reads

$$\begin{aligned} T_n - \text{BE}(^{16}\text{O}) &= E_x - \text{BE}(^{17}\text{O}) \\ E_x &= T_n + \text{BE}(^{17}\text{O}) - \text{BE}(^{16}\text{O}) \approx T_n + 4.15 \text{ MeV}. \end{aligned}$$

The excitation energy can be read from the upper scale in Fig. 5.11. It is straightforward, for instance, to identify the first peak in the figure with the state at 4.56 MeV in ^{17}O having spin and parity $3/2^-$. This peak has a width Γ of about 0.04 MeV. The associated half-life is therefore $\tau = \hbar/\Gamma \approx 10^{-20}$ s. If we compare this time with the interaction time defined above in

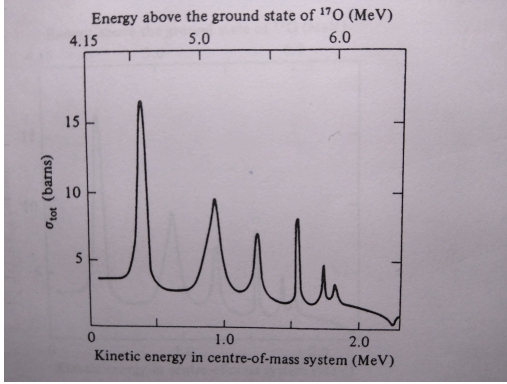


Figure 5.11: Total cross section for neutrons interacting with ^{16}O as a function of the center-of-mass energy. From Cottingham and Greenwood.

Eq. (5.4), we see that this latter is of the order of 10^{-22} s. In other words, the system lives quite longer with respect to the time scale on which it is formed. Therefore, we use time-dependent perturbation theory to describe its formation and subsequent decay.

In this Section we call i , 0 , and f , respectively, the initial channel of the reaction (neutron plus ^{16}O in the example under study), the state that is formed (excited state of ^{17}O), and the final state of the decay. There are, in general, different final states available. The compound nucleus can decay through neutron emission, proton emission (although this hindered by the Coulomb barrier) and γ -decay (this is less likely because it is an electromagnetic decay). f labels a specific final state but, as will be evident in what follows, we often do not detect the specific final state. We will call *inclusive* or *exclusive* cross section the one corresponding to all possible final state decays or the one associated with a specific one. We shall, in what follows, calculate such cross sections. For the sake of simplicity, we shall do it first by neglecting the intrinsic degrees of freedom of the particles, i.e. their spins. Spin factors will be added at the end of the derivation.

The general equation of time-dependent perturbation theory, from quantum mechanics textbooks, is

$$i\hbar\dot{a}_n = \sum_{m \neq n} V_{mn} e^{-i(E_m - E_n)t/\hbar} a_m, \quad (5.79)$$

where m, n are eigenstates of the unperturbed Hamiltonian H_0 (in this case,

the one corresponding to non-interacting neutron and ^{16}O) with corresponding energies E , a labels the probability amplitudes to find the interacting system in these states, and V is the interaction. In the case we study, we can assume a simple Woods-Saxon potential for the neutron-nucleus potential. If we restrict to the states 1 and 0, the previous equation becomes

$$i\hbar\dot{a}_0 = V_{01}e^{-i(E_1-E_0)t/\hbar}a_1. \quad (5.80)$$

In this case $a_0^2 + a_1^2 = 1$ and, while $a_0(t=0) = 0$ and $a_1(t=0) = 1$, we also assume that a_0 will remain quite small and we can approximate a_1 as being 1 for all times. These assumptions, that go together with the more general one of stopping time-dependent perturbation theory at the lowest (i.e., first) order, should be checked *a posteriori*.

We do not want at this stage to consider the possible final states as explicit states. We would like just to consider the possibility that, once in the state 0, the system can decay to the set of states f with a total width Γ that we experimentally can measure. It is a simple exercise to show that this decay can be simply taken into account by changing the latter equation (5.80) into

$$i\hbar\dot{a}_0 = V_{01}e^{-i(E_1-E_0)t/\hbar}a_1 - i\frac{\Gamma}{2}a_0. \quad (5.81)$$

In fact, if we integrate the equation

$$i\hbar\dot{a}_0 = -i\frac{\Gamma}{2}a_0 \quad (5.82)$$

we easily find $a_0 = e^{-\frac{\Gamma}{2\hbar}t}$ that corresponds to $a_0^2 = e^{-t/\tau}$ with $\tau = \hbar/\Gamma$. We then go back and integrate Eq. (5.81). The solution is

$$a_0(t) = V_{01} \frac{e^{-\frac{i}{\hbar}(E_1-E_0)t} - e^{-\frac{\Gamma}{2\hbar}t}}{E_1 - E_0 + i\frac{\Gamma}{2}}. \quad (5.83)$$

The last term in the numerator can be neglected for large values of t . Then,

$$|a_0(t)|^2 = \frac{|V_{01}|^2}{(E_1 - E_0)^2 + \frac{\Gamma^2}{4}}. \quad (5.84)$$

This expression shows already the existence of a resonant-shape excitation probability. We are, nonetheless, interested in the cross section rather than

in the probability of exciting each single state; also, we would like to get rid of the matrix element V_{01} of the potential V , in order to express the result in terms of measurable quantities only. To do so, we consider the full process that includes the decay, $1 \rightarrow 0 \rightarrow f$. The number of events per unit time (i.e., the transition rate R) is equal to the probability amplitude to excite the state 0, divided by the half-life for the process $0 \rightarrow f$:

$$R_{1 \rightarrow 0 \rightarrow f} = |a_0(t)|^2 \frac{1}{\tau_{0 \rightarrow f}} = \frac{|V_{01}|^2}{(E_1 - E_0)^2 + \frac{\Gamma^2}{4}} \frac{\Gamma_f}{\hbar}. \quad (5.85)$$

The trick is now to select the decay back to the initial state 1, that obeys the Fermi Golden Rule

$$\Gamma(E_1) = 2\pi |V_{10}|^2 n(E_1). \quad (5.86)$$

We have preferred to label the state with its energy E_1 rather than simply with 1. One reason is that we need the density of states $n(E_1)$. This can be calculated by assuming that the states of the impinging particle are quantized in a box having volume Ω . An elementary calculations proves that

$$n(E_1) = \frac{\Omega}{(2\pi)^3} 4\pi k^2 \frac{1}{\hbar v}, \quad (5.87)$$

where k is such that $\frac{\hbar k^2}{2\mu} = E_1$ and v is the associated velocity. Finally, the cross section is the transition rate R divided by the incident flux. The flux, with one particle in a box having volume Ω as we have assumed so far, is v/Ω . Therefore, we write

$$\sigma(1 \rightarrow 0 \rightarrow f) = \frac{\Omega}{v} R_{1 \rightarrow 0 \rightarrow f}, \quad (5.88)$$

and we insert here the expression of the transition rate (5.85), with the matrix element $|V_{01}|^2$ written by means of Eq. (5.86) and, then, the density of states $n(E_1)$ taken from Eq. (5.87). The result is

$$\sigma(1 \rightarrow 0 \rightarrow f) = \frac{\pi}{k^2} \frac{\Gamma(E_1)\Gamma_f}{(E_1 - E_0)^2 + \frac{\Gamma^2}{4}}. \quad (5.89)$$

As it has been anticipated, spin factors are missing, so that the previous equation is, strictly speaking, only valid for collisions between two nuclei

having spin zero. It is easy to insert the spin factors, as the cross section is expected to depend on a sum over final states and average over initial states. Therefore, if the initial particles are in the spin states j_1 and j_2 , respectively, and the compound nucleus has spin j , we can write

$$\sigma(1 \rightarrow 0 \rightarrow f) = \frac{\pi}{k^2} \frac{(2j+1)}{(2j_1+1)(2j_2+1)} \frac{\Gamma(E_1)\Gamma_f}{(E_1 - E_0)^2 + \frac{\Gamma^2}{4}}. \quad (5.90)$$

The formula is called *Breit-Wigner*. As we have mentioned, in many instances the exclusive cross section is not measured. The inclusive cross section, or total one, is simply the sum of the cross sections to the various final states. The only factor in (5.90) that depends on f is Γ_f . Since $\sum_f \Gamma_f = \Gamma$, we immediately deduce

$$\sigma(1 \rightarrow 0) = \frac{\pi}{k^2} \frac{(2j+1)}{(2j_1+1)(2j_2+1)} \frac{\Gamma(E_1)\Gamma}{(E_1 - E_0)^2 + \frac{\Gamma^2}{4}}. \quad (5.91)$$

This formula for the inclusive cross section explains indeed Fig. 5.11 above. In this figure one can see several resonances, each of one can be interpreted with the formula (5.91) and the height of the peaks decreases as the factor $\sim \frac{1}{k^2}$ predicts. Although Fig. 5.11 refers to a specific case, neutron resonances exist in all nuclei. The key point is that as the mass number increases the density of such resonances increases. When they overlap, the treatment in terms of independent resonances is no longer valid.

5.7.2 Charged particles reactions

In order to appreciate the difference with the case of the previous subsection, let us assume to form the nucleus ^{17}O in a different reaction like $\alpha + ^{13}\text{C}$. We expect to populate the same states, i.e. to find the same resonances of the compound system. The resonances should be at the same excitation energy in the compound nucleus which implies a different center-of-mass energy, that could be obtained by applying to this case the equation analogous to Eq. (5.79) above. However, the resulting cross section displayed in Fig. 5.12 displays those peaks on top of an exponential background that makes them negligibly small as the energy decreases. We recognize the signature of the Coulomb barrier penetration. In fact, the Coulomb barrier height is in

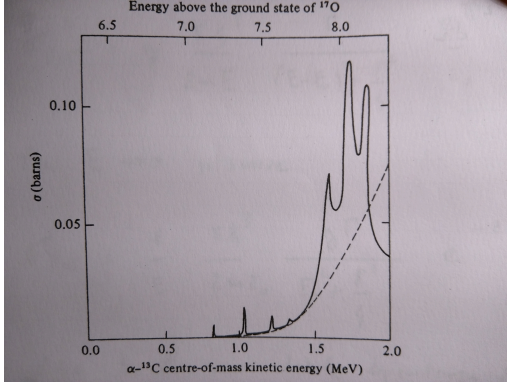


Figure 5.12: Total cross section for α -particles interacting with ^{13}C as a function of the center-of-mass energy. From Cottingham and Greenwood. See the text for a discussion, in particular for an explication of the exponential dashed curve.

general given by

$$V_B = \frac{e^2 Z_A Z_a}{R_A + R_a}, \quad (5.92)$$

and at low energy the charged particle must tunnel this barrier. The probability P for tunneling is invariant under time-reversal, that is, it is the same that has been calculated for the α -decay:

$$\begin{aligned} P &= e^{-2G} \\ G &= \sqrt{\frac{2\mu}{\hbar^2}} \int_a^b dr [V(r) - E_1]^{1/2}, \end{aligned} \quad (5.93)$$

where the barrier is assumed to be spherically symmetric, its radial shape is depicted in Fig. 5.13, a and b are the inner edge of the barrier and the outer edge associated with the incident energy E_1 , respectively, and μ is the reduced mass. In the case of the α -decay, if τ_0 is the typical time interval between successive encounters with the barrier, the transition rate is

$$R = \frac{1}{\tau_0} P = \frac{1}{\tau_0} e^{-2G}, \quad (5.94)$$

and the half-life is then

$$\tau = \tau_0 e^{2G}. \quad (5.95)$$

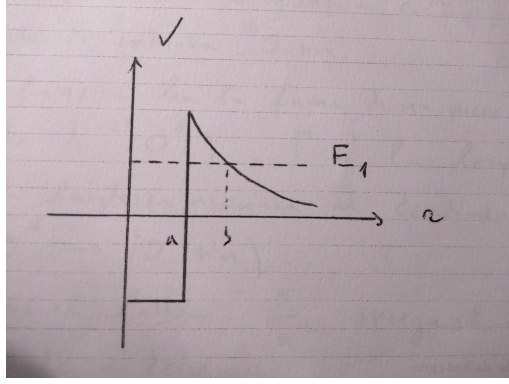


Figure 5.13: Schematic plot for the barrier penetration. See the text for a discussion.

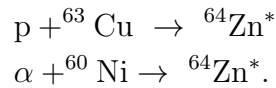
If we identify this with $\hbar/\Gamma(E_1)$ in Eq. (5.91), we obtain from it

$$\sigma(1 \rightarrow 0) = \frac{\pi \hbar}{k^2} \frac{(2j_1 + 1)}{(2j_1 + 1)(2j_2 + 1)} \frac{\Gamma}{(E_1 - E_0)^2 + \frac{\Gamma^2}{4}} \frac{e^{-2G(E_1)}}{\tau_0}, \quad (5.96)$$

where the exponential form has been put in evidence. As $G(E_1)$ is manifestly a decreasing function of E_1 , the cross section exponentially increases as shown in Fig. 5.12. We shall study the case of $E_1 \rightarrow 0$, and evaluate more precisely the factor $G(E)$ in that case, in the following Chapter.

5.7.3 Evidences of the compound nucleus hypothesis and definition of a nuclear temperature

The compound nucleus hypothesis was proposed by Niels Bohr back in 1936. It has been verified much later in many cases. As we stated at the start of the Section, according to this hypothesis the reaction process is slow enough so that the participating nuclei form a compound system in which energy and angular momentum are statistically shared among all nucleons. In this situation, no memory is kept about the initial state. The decay is independent from the initial state. One example is provided by the two reactions



It has been proved that the final decays of the compound nucleus $^{64}\text{Zn}^*$, like $^{63}\text{Zn}+n$, $^{62}\text{Zn}+2n$, $^{62}\text{Cu}+n+p$... have the same cross section in the two reactions provided the compound nucleus is formed at the same excitation energy.

In this respect, the Breit-Wigner formula (5.90) can be factorized as

$$\begin{aligned}\sigma(1 \rightarrow 0 \rightarrow f) &= \sigma(1 \rightarrow 0) B_f \\ B_f &= \frac{\Gamma_f}{\Gamma},\end{aligned}\tag{5.97}$$

where $\Gamma(E_1)$ has been simply written as Γ and B_f is the *branching ratio*. As we said before, we expect that neutron and proton decays are possible only above threshold, but as soon as they are possible $\Gamma_n \gg \Gamma_p \gg \Gamma_\gamma$. In many cases one can approximate Γ with Γ_n , or B_n with 1.

Let us now assume that one can measure the energy spectrum of the emitted neutrons. The assumption of nuclear thermalization is supported by those cases in which the neutrons obey the energy dependence which can be expected in case of a thermalized source, i.e. they obey a Maxwell-Boltzmann distribution like

$$n(E)dE \sim \sqrt{E} e^{-E/T} dE,\tag{5.98}$$

where $n(E)$ is the number of neutrons with energy between E and $E + dE$, the normalization is omitted for the sake of simplicity, and the temperature T is written in units of MeV. Whenever the experimental energy spectra are fitted by a curve of the type (5.98), one can extract a value for T from the fit. Since the compound nucleus excitation energy E is known, the purpose is now to find a relation between E and T .

In a standard thermodynamic picture one would expect

$$\begin{aligned}\frac{1}{T} &= \frac{dS}{dE}, \\ S &= \ln \omega(E),\end{aligned}\tag{5.99}$$

where S is the entropy and $\omega(E)$ is the density of states at the energy E . In principle, it is possible to calculate with modern nuclear structure methods the density of states. A schematic calculation that can be done analytically, albeit with some effort, is based on the Fermi gas model. The key parameter in this case is the single-particle level density at the Fermi surface g_0 . This is the inverse of the gap between the last occupied and first unoccupied state.

We write, in the case of $N = Z = A/2$,

$$\frac{\delta\varepsilon}{\varepsilon} = 2 \frac{\delta k}{k} = \frac{2}{3} \frac{\delta A}{A}, \quad (5.100)$$

where ε is a s.p. energy, $\varepsilon = \frac{\hbar^2 k^2}{2m}$, and the last equality follows if we take k around k_F and we use Eq. (2.27) since of course ρ is proportional to A . The latter equation, in the case $\delta A = 1$ and $\varepsilon = \varepsilon_F$ provides

$$\delta E = \frac{2}{3} \frac{\varepsilon_F}{A} \Rightarrow g_0 = \frac{3}{2} \frac{A}{\varepsilon_F}. \quad (5.101)$$

A parameter which is proportional to the single-particle density of levels around the Fermi energy and which is defined for the sake of praticity, is

$$a = \frac{\pi^2}{6} \frac{3}{2} \frac{A}{\varepsilon_F}. \quad (5.102)$$

At this point, one should count in how many ways particles can be distributed on the single-particle energy levels of the Fermi gas in such a way that the difference between the total energy of the nucleus and that of the g.s. is equal to E . Although tedious this counting can be accomplished and the result is

$$\omega(A, E) = \frac{6^{1/4} g_0}{12(g_0 E)^{5/4}} e^{2\sqrt{aE}}. \quad (5.103)$$

If we use this result, by keeping only the dominant exponential dependence on E , to obtain the temperature from Eq. (5.99) we arrive simply at the equation

$$E = aT^2. \quad (5.104)$$

While the theoretical value in the case of the Fermi gas calculation is $a = A/16 \text{ MeV}^{-1}$, experiments give values of the order of $a = A/8 \text{ MeV}^{-1}$.

5.A Appendices

5.A.1 Transformation between the LAB and COM system for the incident energy

We consider a reaction of the type (5.1) and we assume that A is at rest in the LAB system, while a impinges on it with a kinetic energy $T^{(\text{LAB})} \equiv \frac{1}{2} M_a v_a^2$.

The COM system is defined in terms of the COM coordinate,

$$\vec{r}_{\text{COM}} = \frac{M_A \vec{r}_A + M_a \vec{r}_a}{M_A + M_a}, \quad (5.105)$$

and its velocity,

$$\vec{v}_{\text{COM}} = \frac{M_a \vec{v}_a}{M_A + M_a}. \quad (5.106)$$

The Galilean transformations provide the velocities of A and a in the COM systems, that are

$$\begin{aligned} \vec{v}_a^{(\text{COM})} &= \vec{v}_a - \vec{v}_{\text{COM}} = \frac{M_A \vec{v}_a}{M_a + M_A}, \\ \vec{v}_A^{(\text{COM})} &= -\vec{v}_{\text{COM}} = -\frac{M_a \vec{v}_a}{M_A + M_a}. \end{aligned} \quad (5.107)$$

From this latter equation, the sum of the kinetic energies in the COM frame results as

$$\begin{aligned} \sum_i T_i^{(\text{COM})} &= \frac{1}{2} M_a \frac{M_A^2 v_a^2}{(M_A + M_a)^2} + \frac{1}{2} M_A \frac{M_a^2 v_a^2}{(M_A + M_a)^2} \\ &= \frac{1}{2} \frac{M_A M_a}{M_A + M_a} v_a^2 = \frac{M_A}{M_A + M_a} T^{(\text{LAB})} \approx \frac{A}{A + a} T^{(\text{LAB})}, \end{aligned} \quad (5.108)$$

where in the last step we have approximated the masses with the mass numbers. The quantity $\frac{M_A M_a}{M_A + M_a}$ is the reduced mass μ that will be used throughout the Chapter.

Chapter 6

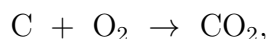
Selected short topics in Nuclear Astrophysics

This Chapter is not intended to give a complete view of the broad domain of Nuclear Astrophysics, but rather to illustrate a few pedagogical cases in which nuclear physics is highly instrumental to tackle and solve astrophysical problems.

6.1 Nuclear reaction in stars: the case of the sun and some generalities

6.1.1 Why solar energy cannot be the outcome of chemical reactions

There are simple arguments that show that the sun cannot produce energy but through nuclear reactions. The solar luminosity, $L_{\odot}$, is known to be about $3.83 \cdot 10^{26}$ W. Let us assume that this is the outcome of usual chemical reactions. As an example, we take the reaction



for which it is known that 1 kg of C provides $3.35 \cdot 10^7$ J. The mass of the sun is known to be $1.99 \cdot 10^{30}$ kg, so if it were made by one third of C and two thirds of O, the total energy available would be

$$E = \frac{1}{3} \cdot 1.99 \cdot 10^{30} \cdot 3.35 \cdot 10^7 \text{ J} = 2.22 \cdot 10^{37} \text{ J}, \quad (6.1)$$

and this can last for a time

$$t = \frac{E}{L_{\odot}} = 1.8 \cdot 10^3 \text{ y.} \quad (6.2)$$

Obviously this time is too short and this was perceived as a great puzzle already back in the 19th century. We know that the sun has an age of the order of 10^9 years and the missing factor 10^6 can be understood, after the discovery of the atomic nucleus, as the scale factor between chemical (i.e. atomic) energies of the order of eV and nuclear energies of the order of MeV.

6.1.2 Hydrostatic equilibrium, average pressure and temperature in the sun

The sun, as other stars, can be thought to be a sphere in hydrostatic equilibrium. We consider the thin region between r and $r + dr$ where a mass $4\pi r^2 dr \rho(r)$ is found if $\rho(r)$ is the density. The gravitational attraction between this mass and the total mass $M(r)$ between zero and r can be written as if $M(r)$ were concentrated in the origin, and reads

$$dF_G = -4\pi G \rho(r) M(r) dr, \quad (6.3)$$

where G is the Newton's constant. If we divide it by the surface $4\pi r^2$, we obtain the gravitational pressure that is directed inwards (as the minus sign in the latter equation implies) and must be balanced by the difference between the pressures on the outer and inner edges of the region, $p(r + dr) - p(r)$. This means

$$\frac{p(r + dr) - p(r)}{dr} = \frac{dp(r)}{dr} = -\frac{dF_G}{4\pi r^2 dr} = -G \frac{M(r) \rho(r)}{r^2}. \quad (6.4)$$

This equation, together with the definition of $M(r)$, that is,

$$M(r) = \int_0^r dr' 4\pi r'^2 \rho(r'), \quad (6.5)$$

could be solved if p as a function of the density were known. Rather than attempting this, we try to extract some average numbers from our empirical knowledge of the solar properties¹. We take Eq. (6.4),

$$\frac{dp}{dr} = -G \frac{M(r) \rho(r)}{r^2},$$

¹The standard solar model (SSM) is described in textbooks and provide a full description of the inner dynamics of the sun.

we multiply both members by $4\pi r^3$ and integrate from zero to the radius R which is defined as the point where the pressure becomes zero. This amounts to writing

$$\int_0^R 4\pi r^3 \frac{dp}{dr} dr = -G \int_0^R \frac{M(r)\rho(r)}{r} 4\pi r^2 dr. \quad (6.6)$$

The r.h.s. is simply the gravitational energy while the l.h.s can be integrated by parts to give

$$4\pi r^3 p(r)|_0^R - 3 \int_0^R p(r) 4\pi r^2 dr = -3\bar{p}V, \quad (6.7)$$

where the first term is manifestly zero while the second one defines an average pressure $\bar{p}$. From the latter equation, and knowing the radius of the sun which is $6.96 \cdot 10^8$ m, one extracts an average pressure of $5.35 \cdot 10^{13}$ Pa. We can also extract an average temperature by means of the perfect gas equation. If $\bar{m}$ is an average mass of the constituents², then

$$\begin{aligned} \bar{p} &= \frac{Nk_B T}{V} = \frac{N\bar{m}}{V} \frac{1}{\bar{m}} k_B T = \frac{\bar{\rho}}{\bar{m}} k_B T, \\ k_B T &= \frac{\bar{p}\bar{m} \frac{4}{3}\pi R_\odot^3}{M_\odot} \approx 0.2 \text{ keV}, \end{aligned} \quad (6.8)$$

where k_B is the Boltzmann's constant and $\bar{\rho}$ is the average density. The value of $k_B T$, although it corresponds to a rather high temperature on the terrestrial scale (several millions of K), is quite small on the nuclear scale. In this regime, nuclear reactions are very much below the barrier and are extremely hindered. This explains the long life of the sun (and of the life on the Earth, by the way).

6.1.3 Brief account of solar reactions in the sun

Bethe and Cricthfield were the first to propose a cycle of reactions that is able to turn H into He. Nowadays, this is known as pp1-cycle. Bethe is also responsible for pointing out the existence if the so-called CNO-cycle, and these discoveries were awarded with the Nobel prize. The pp-cycle reactions are depicted in Fig. 6.1 while the CNO cycle is described in specialized textbooks.

²The sun is made with about 71% of H, 27% of He and small amount of heavier elements. If only H were present, the average mass of one proton and one electron is ≈ 0.5 amu. Including He we can write $\bar{m} \approx 0.61$ amu.

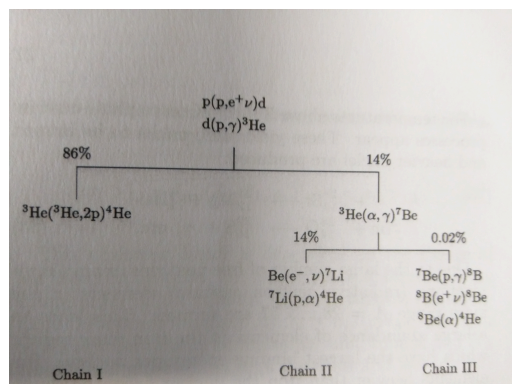


Figure 6.1: Main cycles that contribute to H burning, that is, transmutations of H into He, in the sun.

6.1.4 The case of other stars

The main observable that determines the properties of a star is its mass. As is evident from our discussion about the sun in subsec. 6.1.2, had we taken a star with larger mass we would have found larger values of T and p . In fact the sun belongs to the low-mass stars (mass equal to $0.4\text{--}2 M_\odot$). It is relatively young and at the present stage it still converts H into He as we have just described in subsec. 6.1.3. Such stars are prone to exhaust H, contract themselves, increase the values of T , and start burning He. Eventually, they may end up in a white dwarf made up with mainly C and O.

Massive stars behave quite differently. If we take, as an extreme case, a star that starts from a mass around $25 M_\odot$, after burning H and He it can further contract due to gravitational forces, increase internal temperature and pressure, and start ignition of heavier elements; thus, such stars go through C-, Ne-, O-, and Si-burning. We do not discuss the details of the nuclear reactions that take place, nor we estimate the time scales and the amount of energy released.

The only point we want to make here is that in other stars than the sun, obviously different reactions can take place and the temperature can be higher. In the next Section we shall discuss how to estimate the reaction rates of these reactions in a general fashion, once astrophysical considerations have provided a reliable number for the temperature.

After Si is exhausted, most of the star is composed of nuclei like ^{56}Fe . Then, no further energy can be released via fusion reactions and the equil-

brium between gravitational energy, that tends to contract the system, and energy generated by nuclear processes that tend to expand the system, breaks down. The so-called core collapses and the electron degeneracy pressure is able to counteract this contraction only when the mass is smaller than the so-called Chandrasekhar limit ($\approx 1.4 M_\odot$). The core collapse is accelerated by electron capture by iron nuclei (the so-called neutronization), and actually by other complex phenomena that reduce the available energy density and pressure. When the core reaches such a high density that neutrons feel the short-range hard-core repulsion (that we have described in Chapter 1), the core bounces back and a shock wave produces the so-called supernova explosion. How this happens in detail, is not yet well understood.

6.2 The Gamow peak

Let us take a generic reaction $k \rightarrow j$. The cross section will depend strongly on energy because it is associated with a barrier penetration. The energy under exam is, in turn, related to the relative velocity between k and j , $v = |\vec{v}_k - \vec{v}_j|$, by $E = \frac{1}{2}\mu v^2$. At a given energy E , the cross section is the number of reactions per unit time [i.e. the transition rate $r(E)$] and per target nucleus, divided by the incident flux. We call d^3n_k and d^3n_j the number of particles of the species k and j with velocities in a three-dimensional infinitesimal volume in the phase space around $\vec{v}_k$ and $\vec{v}_j$, respectively. Then, we can write

$$\sigma(E) = \frac{r(E)/d^3n_j}{d^3n_k v}, \quad (6.9)$$

while the total reaction rate $r_{k \rightarrow j}$ integrated over all possible velocities of the particles k and j reads

$$r_{i \rightarrow j} = \int r(E) = \int \sigma(E) v d^3n_j d^3n_k. \quad (6.10)$$

This reaction rate can be calculated by assuming a Maxwell-Boltzmann distribution for the particle velocities, namely

$$d^3n_j = n_j \left(\frac{m_j}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{m_j v_j^2}{2k_B T} \right) d^3v_j, \quad (6.11)$$

where n_j is the total number of particles j , m_j is their mass, and T is the temperature of the environment in which the reaction takes place. Exactly

the same equation holds for d^3n_k . It is convenient to change variables to the center-of-mass and relative coordinates,

$$\begin{cases} \vec{R} = \frac{m_j \vec{r}_j + m_k \vec{r}_k}{m_j + m_k}, \\ \vec{r} = \vec{r}_j - \vec{r}_k \end{cases} \quad (6.12)$$

[this latter equation is a generalisation of Eq. (1.4)]. Then, velocities must be transformed accordingly and the following well-known result can be found

$$\frac{1}{2}m_j v_j^2 + \frac{1}{2}m_k v_k^2 = \frac{1}{2}M V^2 + \frac{1}{2}\mu v^2 = E_{\text{COM}} + E, \quad (6.13)$$

where M is the total mass and $\vec{V}$ is the velocity associated with $\vec{R}$. With this choice,

$$r_{i \rightarrow j} = n_j n_k \frac{m_j^{3/2} m_k^{3/2}}{(2\pi k_B T)^3} \int \sigma(E) v e^{-\frac{1}{k_B T}(\frac{1}{2}M V^2 + \frac{1}{2}\mu v^2)} d^3V d^3v. \quad (6.14)$$

The integral over the center-of-mass velocity $\vec{V}$ can be done straightforwardly since the cross section does not depend on it, and one is left with a standard Gaussian integration. The result is

$$r_{i \rightarrow k} = n_j n_k \frac{\mu^{3/2}}{(2\pi k_B T)^{3/2}} \int \sigma(E) v e^{-\frac{\mu v^2}{2k_B T}} 4\pi v^2 dv. \quad (6.15)$$

We change the integration variable to $E = \frac{1}{2}\mu v^2$ and obtain our final result

$$r_{i \rightarrow j} = n_j n_k \left(\frac{8}{\pi \mu} \right)^{1/2} (k_B T)^{-3/2} \int dE \sigma(E) E e^{-E/k_B T}. \quad (6.16)$$

This latter equation would imply that one has to know the cross section at incident energies between 0 and ∞ in order to extract the reaction rate. However, this is not the case: we shall demonstrate that only a limited range of incident energies must be actually explored. The cross section $\sigma(E)$, because of the Coulomb barrier, is an exponentially growing function of E whereas the exponential (Boltzmann) factor is decreasing with E . There will be a limited region around the maximum of the integrand where this is non negligible. The maximum of the product between the two exponential

factors $\sigma(E)$ and $e^{-E/k_B T}$ is called the *Gamow peak*. The linear function E is negligible in the present context. We calculate the position of the Gamow peak by starting from Eq. (5.96). As anticipated at that stage, we first evaluate the Gamow factor $G(E)$ at extremely low energy.

We re-write the cross section that we have already written in Eq. (5.96) as

$$\sigma(E) = \frac{1}{E} S(E) e^{-2G(E)}, \quad (6.17)$$

where the dominant exponential factor has been kept, the quantity $1/E$ corresponds to the factor $1/k^2$ in Eq. (5.96), and the other factors, either constants or factors that have a mild dependence on E , have been gathered under the symbol $S(E)$. We remind that the Gamow factor, already defined in Eq. (5.93), reads

$$G(E) = \sqrt{\frac{2\mu}{\hbar^2}} \int_a^b dr [V(r) - E]^{1/2},$$

and the points a and b are defined in Fig. 5.13. Let us identify a with the nuclear radius, and b with the classical turning point r_C , that is defined through

$$E = V(r_C) = \frac{Z_i Z_j e^2}{r_C} \quad \Rightarrow \quad r_C = \frac{Z_i Z_j e^2}{E}. \quad (6.18)$$

In such a case, the integral appearing in the Gamow factor can be evaluated analytically with the replacement $r = r_C \cos^2 \theta$. The result is

$$G(E) = \frac{r_C}{\hbar} \sqrt{2\mu E} \left[\cos^{-1} \sqrt{\frac{R}{r_C}} - \sqrt{\frac{R}{r_C} \left(1 - \frac{R}{r_C} \right)} \right]. \quad (6.19)$$

For energies in the range between fractions of keV [like in Eq. (6.8)] and 10 keV the assumption $r_C \gg R$ is extremely appropriate. Then, we can approximate well

$$G(E) = \frac{r_C}{\hbar} \frac{\pi}{2} \sqrt{2\mu E}. \quad (6.20)$$

We derive r_C from Eq. (6.18) and we consider that $e^2 = \alpha \hbar c$ where α is the fine structure constant. Then

$$e^{-2G(E)} = \exp \left[-\frac{\pi \alpha Z_i Z_j \sqrt{2\mu c^2}}{\sqrt{E}} \right] \equiv e^{-b/\sqrt{E}}, \quad (6.21)$$

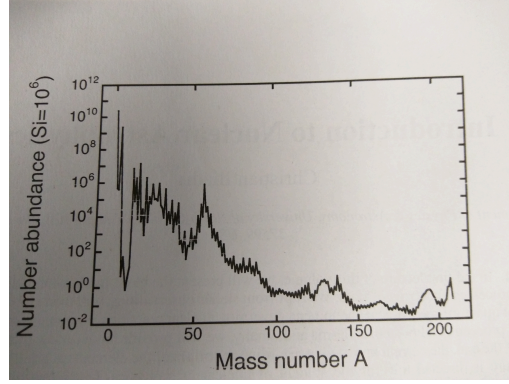


Figure 6.2: Abundance of the elements in the solar system, normalized to the number of Si atoms which is taken as 10^6 . From Iliadis.

and, thus, the integrand of Eq. (6.16) behaves as

$$f(E) = \exp \left[-\frac{E}{k_B T} - \frac{b}{\sqrt{E}} \right] \quad (6.22)$$

if we just neglect $S(E)$. We define E_0 as the maximum of the function $f(E)$. A simple maximization of this function yields

$$E_0 = \left(\frac{b k_B T}{2} \right)^{2/3} = 1.22 \left(Z_i^2 Z_j^2 A_{\text{red}} T_6^2 \right)^{1/3} \text{ keV}, \quad (6.23)$$

where in the last step we have defined the reduced A equal to $A_i A_j / (A_i + A_j)$ and the quantity T_6 which is the temperature in units of 10^6 K. The opportune constants have been grouped. E_0 is the energy of the Gamow peak.

6.3 The nucleosynthesis problem

After having laid out the main features of reactions in stars, and especially the fact that these reactions take place at low energies around the Gamow peak, many further questions arise. How do these reactions have produced, and keep producing, the elements that we find in the Universe ? This is perhaps the main question and it has been listed among the most crucial unanswered problems in physics.

Fig. 6.2 shows the relative abundances of elements in the solar system. H and He are the most abundant and their formation happened during the

Big Bang. There is a big drop in abundances in correspondence with Li, Be and B, and a sharp increase at C and O (which explains, among the rest, the existence of life on the Earth). After that, the curve on the average decreases but there is a huge peak around Fe, Co and Ni which is referred to as the iron peak, plus other peaks that can be connected with nuclei having magic or double-magic character. These peaks are then associated with those nuclei which have the largest binding energies and must be the result of nuclear processes that tend to maximize their production. The reactions among charged-particles, and the associated reaction rates defined by Eq. (6.16), can explain the productions up to the iron peak. To explain the production of heavier elements, that possess a neutron excess, neutron capture processes need to be invoked.

In usual conditions, the neutron density is not very large and the neutron capture is slowed down by β -decay. A nucleosynthesis process that takes place in such conditions is called the s-process (s stands for slow). Only much larger neutron densities can give rise to the so-called rapid, or r-process. In such a case, the neutron capture rates is so high that the process can proceed to more neutron-rich systems without being hindered by β -decays.

Reaction network equations, in which the number of particles i as a function of time is written in terms of the reaction rates that create or destroy them, can be devised. One of the difficulties is identifying the appropriate astrophysical environment. The exact site of the r-process is for instance not yet known with certitude.

6.4 Neutron stars

The existence of systems made up by only neutrons and bound by the gravitational force was predicted by Landau long time ago, at the beginning of the 1930s. The discovery of neutron stars came only much later, in 1967.

It can be, at our level, understood by means of a simple exercise based on the mass formula,

$$\text{BE}(A, Z) = a_V A - a_S A^{2/3} - a_A \frac{(N - Z)^2}{A} + \text{pairing term}.$$

A system with only neutrons has no Coulomb, and is expected to be very large if the binding energy of the latter formula, which is negative, must be counterbalanced by the gravitational energy $\frac{3}{5} \frac{GM^2}{R}$, where G is the Newton's

constant, and M, R are the mass and radius of the system. Being the system large, we can neglect the surface energy and obviously the pairing contribution. We arrive at the following balance between nuclear and gravitational energies, namely

$$a_V A - a_A A + \frac{3}{5} \frac{Gm^2}{r_0} A^{5/3} = 0, \quad (6.24)$$

where m is the nucleon mass and the radius is taken as usual from Eq. (2.1). If we replace some reasonable numbers for the parameters coming from the mass formula, like $a_V = 15.85$ MeV and $a_A = 23.21$ MeV, we obtain

$$A \approx 5 \cdot 10^{55}. \quad (6.25)$$

The corresponding mass and radius are

$$M = mA \approx 10^{29} \text{ kg} \approx 0.05 M_\odot \quad R = r_0 A^{1/3} \approx 5 \text{ km}. \quad (6.26)$$

These estimates have the qualitative importance to show that a neutron star may exist but turn out to be rather crude. The real mass of a neutron star is between 1 and 2 solar masses, and the radius around 10 km. $1.4M_\odot$ is a kind of canonical mass that characterise many observed stars. The number of particles is larger than what predicted by the latter equation and is rather around 10^{57} . Most of them are neutrons but there are protons and electrons, whose number can be fixed once the number of neutrons is given, through the two conditions of charge neutrality and equilibrium with respect to weak processes (appropriate relation between chemical potentials). The error in the estimates comes mainly from the assumption of a constant density. We know to some extent nowadays that while in the inner part the density can reach several times the nuclear saturation density, this number goes down to fractions of the saturation density before the surface of the star is reached.

The calculation of the mass and radius of a neutron star can be performed by assuming hydrostatic equilibrium. The classical (i.e. Newtonian) version of this assumption has been elucidated in 6.1.2 and led to Eq. (6.4). Actually neutron stars are so compact (that is, the local mass density is so large) that corrections from general relativity cannot be neglected. We cannot discuss here how from the general equations of general relativity one can deduce, with assumptions that include the spherical symmetry, the corrections to Eq. (6.4) and we merely write down the result, which is the celebrated

Tolman-Oppenheimer-Volkov (TOV) equation:

$$\frac{dp(r)}{dr} = -G \frac{M(r)\rho(r)}{r^2} \left(1 + \frac{p}{\rho c^2}\right) \left(1 + \frac{4\pi p r^3}{M(r)c^2}\right) \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}, \quad (6.27)$$

where now ρ is the energy density $\mathcal{E}$ divided by c^2 . In the case of the Newtonian version of this equation we can replace this with the mass density. We remind that the pressure p can be related to the energy density (or to the total energy E or the energy per particle E/A)

$$p = -\frac{\partial E}{\partial \Omega} = \rho \frac{d\mathcal{E}}{d\rho} = \rho^2 \frac{d}{d\rho} \frac{E}{A}. \quad (6.28)$$

Therefore, if we have at our disposal a nuclear model that provides the energy as a function of the density, we can insert this in the TOV equation (6.27), solve it, and obtain a density, mass and pressure profile for the star. The solution will depend on one boundary condition since the differential equation is of the first order. A mass-radius profile will be obtained, depending on the chosen value of the central pressure.

There are many interesting phenomena in neutron stars. They are characterised by huge magnetic fields, they rotate and the angular period displays interesting time variations, they cool down as a function of time since they are hotter as soon as they are formed (proto neutron stars). We do not discuss such issues, neither we deal with the question whether non-nucleonic particles can exist in the inner part of neutron stars.

Chapter 7

Fusion and fission reactions

7.1 Heavy-ion fusion reactions

We do not treat here the hydrogen fusion, or anyway the case of light nuclei. We are interested in the case of heavy ions. Fusion is often the only method, starting from two colliding nuclei, to form a high-spin rotating system - if one is interested in studying rotations, or other kinds of excited states that can be further populated through γ and neutron emission. Fusion is also an important reaction in stellar environments (cf. the discussion in Chapter 6). Last but not least, as we will see in this Section, fusion at energies below the barrier is a unique probe of nuclear structure properties.

Let us assume we want to study the fusion between two ions (A_1, Z_1) and (A_2, Z_2) . Fusion is classically possible if the potential between the ions can be overcome, but is also possible below any potential barrier because nuclei are quantal objects. The problem can be treated by separating different partial waves characterised by the angular momentum l ; the effective potential between the ion reads

$$V_l(r) = V_{Nuclear}(r) + V_{Coulomb}(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2}, \quad (7.1)$$

where the nuclear potential $V_{Nuclear}$ can be taken with a Woods-Saxon shape, the Coulomb potential $V_{Coulomb}$ can be approximated as that of two uniformly charged spheres, and the last term is the well-known centrifugal potential, μ being the reduced mass. By choosing the example of ^{16}O and ^{154}Sm we find the behaviour that is displayed in Fig. 7.1 (in the figure caption we mention the parameters that have been used), as far as the nuclear

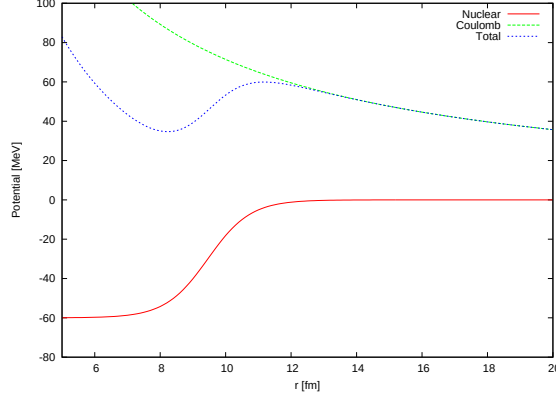


Figure 7.1: Potential among the two nuclei ^{16}O and ^{154}Sm . The nuclear potential has the Woods-Saxon shape (2.3) with radius R given by $1.2 \left(A_1^{1/3} + A_2^{1/3} \right)$ [fm] and depth equal to 60 MeV, while the Coulomb potential is that of two point charges Z_1 and Z_2 . Taken from K. Hagino.

and Coulomb term are concerned. We now move on and consider the role of the angular momentum of the relative motion. We start from a simple classical approximation in which the relationship between the impact parameter b and the angular momentum l is simply

$$l = kb, \quad (7.2)$$

where $E = \frac{\hbar^2 k^2}{2\mu}$.

We define a *grazing* angular momentum l_g in the following way: the system is characterised by this angular momentum if the corresponding energy E is equal to the potential at its maximum. If R is the radial value corresponding to this maximum, then l_g is the solution of the implicit equation

$$E = V_{l_g}(R) = V_{\text{Nuclear}}(R) + V_{\text{Coulomb}}(R) + \frac{\hbar^2 l_g(l_g + 1)}{2\mu R^2}. \quad (7.3)$$

In order to have a simple estimate of this solution, we employ Eq. (7.2) and define accordingly $b_g = l_g/k$. We can then write

$$E = V + \frac{\hbar^2 (kb_g)^2}{2\mu R^2}, \quad (7.4)$$

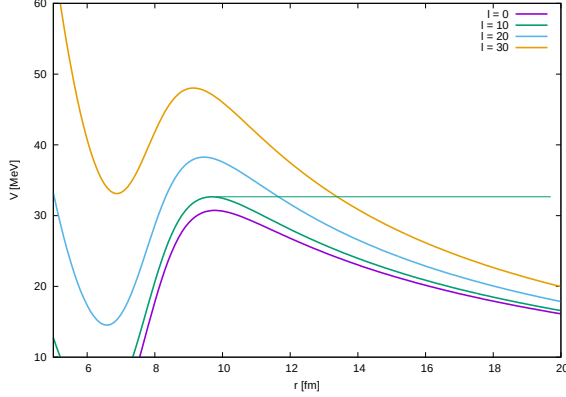


Figure 7.2: Potential among the two nuclei ^{16}O and ^{58}Ni . In this case, only the total effective potential is displayed, including the centrifugal one, for different values of l . If the energy is the one indicated by the horizontal line, the meaning of the approximation that fusion takes place when $l > l_g$ which is equal to 10 in this case, becomes evident. Taken from K. Hagino.

where V is a shorthand notation for the sum of the nuclear and Coulomb potentials. We can see such simple estimate in Fig. 7.2. As we discuss in the next subsection, the correspondingly simple estimate for the fusion cross section consists in assuming that as soon as the impact parameter is equal to, or smaller than, the grazing one b_g then fusion takes place.

7.1.1 Classical approximation

In the purely classical approximation, fusion takes place in all those events in which the impact parameter b is smaller than the grazing one b_g .

$$\sigma^{\text{cl.}} = \int_0^{b_g} 2\pi b \, db = \pi b_g^2. \quad (7.5)$$

We take b_g from Eq. (7.4) and we also use the fact that $E = \frac{\hbar k^2}{2\mu}$. With simple manipulations we arrive at

$$\sigma^{\text{cl.}} = \pi R^2 \left(1 - \frac{V}{E} \right), \quad (7.6)$$

where V is the barrier height, that is the maximum value of the potential V , and R is its radial position. Data taken at different energies above the barrier obey such law and interpolation allows extracting the values of V and R . As soon as we go below the barrier we must take into account the penetration factor.

7.1.2 Sub-barrier fusion and Wong's formula

The correspondence between l and b that has been approximately described by Eq. (7.2) should be replaced by

$$l + \frac{1}{2} = kb. \quad (7.7)$$

The difference with the previous estimate is not relevant for large values of kb . The rationale behind Eq. (7.7) is that this relation corresponds to the maximum possible value for the reaction cross section (cf. the Appendix). Then, the classical value (7.5) can be written in terms of the angular momentum l and complemented by a transmission coefficient for each partial wave $P_l(E)$, that gives precisely the probability that such partial wave penetrates the potential barrier,

$$\sigma = \int 2\pi b \, db = \frac{\pi}{k^2} \sum_l (2l+1) \rightarrow \frac{\pi}{k^2} \sum_l (2l+1) P_l(E). \quad (7.8)$$

Since we may use this formula around the barrier, the best approximation for the transmission coefficient is

$$P_l(E) = [1 + e^{2G(E)}]^{-1}, \quad (7.9)$$

with

$$G(E) = \sqrt{\frac{2\mu}{\hbar^2}} \int_a^b dr \left[V + \frac{\hbar^2 (l + \frac{1}{2})^2}{2\mu r^2} - E \right]^{1/2} \quad (7.10)$$

[a and b are the inner and outer edge of the barrier, as in Eq. (5.93)]. Note that the previous formula reduces to $P(E) = e^{-2G(E)}$ exactly in the form of Eq. (5.93) if we are well below the barrier and the Gamow factor $G(E)$ is large.

There is a way to evaluate analytically the Gamow factor if one approximates the shape of the barrier by using a reversed parabola, that is, with the replacement

$$V(r) + \frac{\hbar^2 \left(l + \frac{1}{2}\right)^2}{2\mu r^2} \rightarrow V_B - \frac{1}{2}\mu\omega^2 (r - R)^2, \quad (7.11)$$

where V_B is the maximum of the barrier, R is its radial position as it was above, and ω is a parameter to be fitted to reproduce the barrier shape at best (at least close to the maximum). Then, the Gamow factor becomes

$$\begin{aligned} G(E) &= \sqrt{\frac{2\mu}{\hbar^2}} \int_a^b dr \left[V_B - E - \frac{1}{2}\mu\omega^2 (r - R)^2 \right]^{1/2} \\ &= \frac{\pi}{\hbar\omega} (V_B - E). \end{aligned} \quad (7.12)$$

Eventually,

$$\begin{aligned} P_l &= \left[1 + e^{-\frac{2\pi}{\hbar\omega} (V_B - E)} \right]^{-1} \\ \sigma &= \frac{\pi}{k^2} \sum_l (2l + 1) \frac{1}{1 + \exp \left\{ -\frac{2\pi}{\hbar\omega} \left[V_{B0} + \frac{\hbar^2 (l + \frac{1}{2})^2}{2\mu R^2} - E \right] \right\}}, \end{aligned} \quad (7.13)$$

where V_{B0} is the barrier height in the case of $l = 0$. The sum in the latter equation can be replaced by an integral and solved, and the result is the famous Wong's formula, that is,

$$\sigma = \frac{\hbar\omega R^2}{2E} \ln \left[1 + e^{\frac{2\pi}{\hbar\omega} (E - V_{B0})} \right]. \quad (7.14)$$

It is easy to show that this formula reduces to the classical one, Eq. (7.6), in the case $E \gg V$ that is far above the barrier.

7.1.3 Failure of potential models below the barrier: fusion enhancement

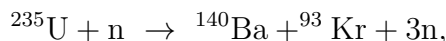
The models based on a potential barrier penetration works relatively well for light systems, but underpredict the fusion cross sections for heavy systems. In those cases, sub-barrier fusion displays a larger cross section than what is

predicted by any static potential model. This has been pointed out around the turn of the 1970s and during the 1980s. The origin could not be traced back to the choice of the specific potential. It is due instead to dynamical effects. This goes beyond our scope here.

7.2 Nuclear fission

7.2.1 General aspects

After the discovery of the neutron, in 1932, there was immediately much interest in the possible reactions that this new projectile, unhindered by the Coulomb barrier, could induce. One of the directions was the exploration whether neutron capture could lead to the production of elements with atomic number Z larger than 92. There were difficulties, at that time, to identify the final products of neutron-induced reactions. O. Hahn and F. Strassmann showed in 1939 that elements with mass equal roughly to one half of U were produced, when neutrons impinged on that target. L. Meitner and O. Frisch gave the correct interpretation of the phenomenon which is now called *fission*. A typical fission process is



and this reaction is highly exoenergetic since it liberates about 200 MeV.

The first question is how fission occurs, why it is not spontaneous for U and needs to be induced by neutrons, why it is spontaneous for some heavier elements, and why it simply cannot occur for light ones. In order to eventually undergo fission, the nucleus has to go through shapes like those that are schematically drawn in the left part of Fig. 7.3. The deformation parameter can be better defined by recalling the general formula for deformed shapes, Eq. (4.1), and specializing it to the case of axial symmetry, in which one has

$$R(\theta) = R [1 + \alpha'_2 P_2(\cos\theta) + \alpha'_3 P_3(\cos\theta) + \dots], \quad (7.15)$$

where R_0 has been written as R for the sake of simplicity, and the parameters α' are related to α in Eq. (4.1) by simple relationships since

$$Y_{l0}(\theta) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta).$$

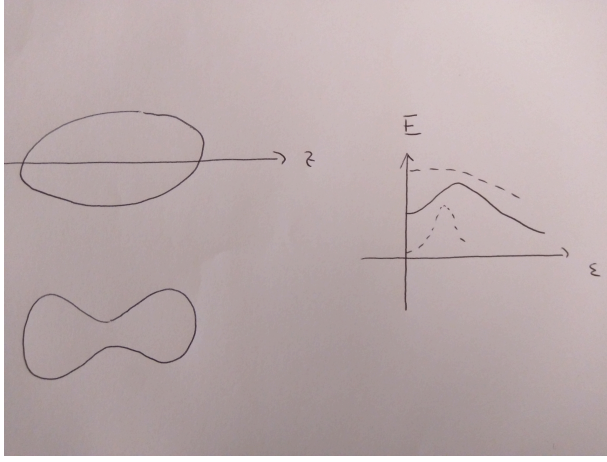


Figure 7.3: In a very rough way, we sketch in the left part of the figure the elongated shapes through which the nucleus is supposed to evolve in order to undergo fission. As a function of the distance between the two centers, which is proportional to the deformation parameter ϵ defined in the main text, we sketch on the right the trend of the total energy. See the text for further discussion.

Although in realistic cases one has to take into account all possible deformations up to multipole 3, 4 or even higher, and also triaxial shapes, for pedagogical reasons we discuss briefly only what happens if we stop at quadrupole. We write α'_2 as ϵ . By knowing that $P_2 = \frac{1}{2}(3\cos^2\theta - 1)$ one can deduce that

$$\begin{aligned} R_z &= R(1 + \epsilon), \\ R_\perp &= R\left(1 - \frac{\epsilon}{2}\right) \approx R\frac{1}{\sqrt{1 + \epsilon}}, \end{aligned} \quad (7.16)$$

where we have assumed that ϵ is small to describe only the first part of the fission process. In a microscopic approach, one should calculate the total energy from a state-of-the-art theory, but we shall use simply the energy of the semi-empirical mass formula. By changing the shape according to Eq. (7.16), we can calculate both the change in surface energy and in Coulomb energy. The other terms of the mass formula do not produce any variation of the total energy if the volume is unchanged. The surface energy is taken as σS , where S is the surface and σ is the surface tension. The surface of the ellipsoid defined by Eq (7.16) can be calculated to be $4\pi R^2 \left(1 + \frac{2}{5}\epsilon^2\right)$, so

that

$$E_{\text{surf}} = E_{\text{surf},0} \left(1 + \frac{2}{5} \epsilon^2 \right), \quad (7.17)$$

where $E_{\text{surf},0}$ is the surface energy of the sphere having radius R . In a similar way, one can calculate the variation of the Coulomb energy with elementary geometry and obtain

$$E_{\text{Coul}} = E_{\text{Coul},0} \left(1 - \frac{1}{5} \epsilon^2 \right). \quad (7.18)$$

The net result is that the variation of the energy for small values of ϵ is expected to be given by

$$\frac{\Delta E}{\epsilon^2} = \frac{2}{5} a_{\text{S}} A^{2/3} - \frac{1}{5} a_{\text{C}} Z A^{-1/3}. \quad (7.19)$$

This quantity can be evaluated by using the typical parameters of the mass formula. It is zero if

$$\frac{Z^2}{A} = \frac{2a_{\text{S}}}{a_{\text{C}}} \approx 51. \quad (7.20)$$

This explains why only very heavy elements can undergo spontaneous fission. The parameter Z^2/A is often called *fissility parameter*. If it is of the order of, or larger than, the value given in Eq. (7.20) then fission is spontaneous. This is the situation depicted by means of the highest dashed line in the right part of Fig. 7.3. In the case of the full line, ΔE is positive and there is a fission barrier. Spontaneous fission is still possible but the associated lifetime can be large. The lowest dashed line represents a light nucleus where the lifetime of induced fission becomes essentially infinite. In the case of induced fission, a neutron is captured and the initial energy of the system is larger than the minimum of the full line in Fig. 7.3, at $\epsilon = 0$. If the initial energy is larger than the fission barrier, the process is spontaneous. In fact, fission can also be induced in other ways, e.g. by photon absorption. Some barriers are well known. For instance, in ^{236}U , the barrier height is 5.3 MeV while by capturing a very low energy neutron the energy of ^{235}U becomes 6.55 MeV. On the other hand, in ^{239}U the barrier height is 5.5 MeV and the energy of ^{238}U plus a thermal neutron is 4.81 MeV. This is why the former is a fissile material while the second is not.

Modern sophisticated calculations of fission barriers exist. In fact, one should not only be able to predict the fission barrier but also the type of fragments and their kinetic energies. While several data exist, model predictions

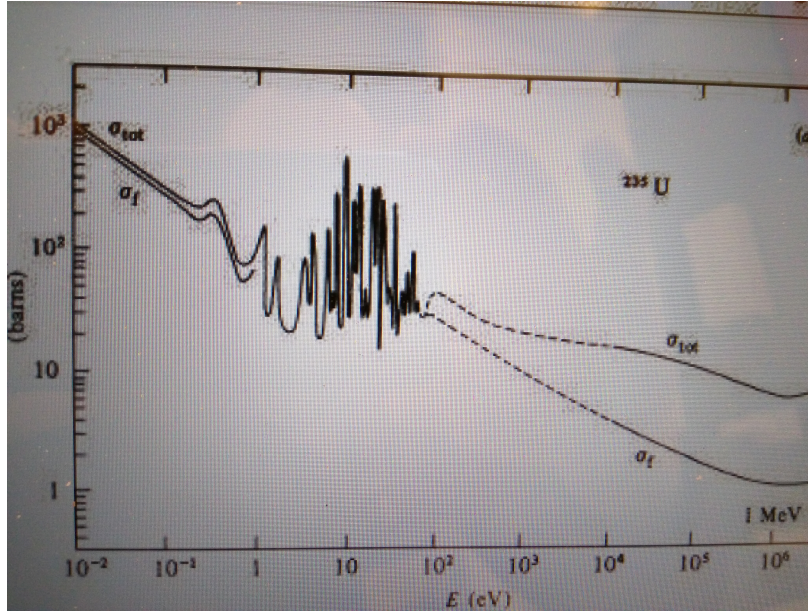


Figure 7.4: Cross section of neutrons on ^{235}U as a function of the neutron energy. σ_{tot} is the total cross section, while σ_f is the cross section for fission. Taken from Cottingham and Greenwood.

are not always reliable. Some general features are known. In case of fission induced by very low-energy projectiles, the distribution of fission fragments turns out to be asymmetric. Such distribution evolves towards being more symmetric in the case of fission induced by projectiles with higher energies. Ternary fission also exists. We do not discuss these issues, that can be found in the more specialized literature.

7.2.2 The fission of uranium and the possibility of a chain reaction

As we have to some extent anticipated in the previous subsection, U fission presents special interest. A low-energy neutron impinging on ^{235}U produces fission with a release of 200 MeV of energy. This is of course relevant for energy production. The ratio of energy released vs. mass of fuel, $E/m_{\text{fuel}}c^2$ is then of the order of 10^{-3} . We can compare this number with a typical one associated with the release from chemical reactions. In Chapter 6 (Sec.

6.1.1) we have considered C and pointed out that 1 kg of C produces $E = 3.35 \cdot 10^7$ J, and in this case the ratio is 10^{-9} . Every fission event produces 2-3 neutrons. Typically, 167 MeV go into kinetic energy of the fragments, 5 MeV go into kinetic energy of the neutrons, and 6 MeV are found in γ -rays. This is the energy which is actually available. About 15 MeV are the result of β -decay, either in terms of electron energy or delayed γ -rays, and this energy may appear with a large time delay, even years. Finally, 12 MeV are under the form of neutrinos and this energy is also escaping.

The energy release is not the only keypoint. If one wishes to produce energy, the process must have a large cross section and must be self-sustained. The ≈ 2.5 neutrons produced in each reaction can ignite a chain process under given circumstances as we explain below. As for the cross section, we show in Fig. 7.4 the total cross section of neutrons on ^{235}U . For thermal neutrons, below 0.1 eV, this cross sections reaches 10^3 b, which is a huge value, but it drops to about 7 b in the MeV range. As we said, one is mainly interested in the fission cross section, which is about 84% of the total one at very low energy, while in the MeV range fission takes place about 1 time over 6.

We briefly discuss how these numbers impact the estimate of a chain reaction. The typical density of a U sample is $4.8 \cdot 10^{28} \text{ m}^{-3}$. Most of this material is made of ^{238}U but we assume a 100% ^{235}U enriched sample just for the sake of illustration. The reaction rate for one neutron on U is $\rho v \sigma$ where v is in this case the neutron velocity. Thus, the time between two reactions is

$$\tau = \frac{1}{\rho v \sigma}, \quad (7.21)$$

and the mean free path is simply

$$l = \frac{1}{\rho \sigma} = \frac{1}{4.8 \cdot 10^{28} \cdot 7 \cdot 10^{-28}} \text{ m} \approx 3 \text{ cm}, \quad (7.22)$$

where we have taken the value of σ for energies in the MeV range according to the numbers given in the previous paragraph, that is, because we expect 2.5 neutrons for each reaction carrying away 5 MeV all together. With an energy of 2 MeV, the velocity of the neutron is $2 \cdot 10^7$ m/s and the time τ is about $1.5 \cdot 10^{-9}$ s.

Let us define as $n(t)$ the neutron number at time t . This quantity will obey the following equation,

$$n(t + dt) = n(t) + (\nu q - 1) \frac{dt}{t_0} n(t), \quad (7.23)$$

7.A. APPENDIX: RELATION BETWEEN IMPACT PARAMETER AND ANGULAR MOMENTUM

where the second term describes the increase due to new fission processes. We assume that each fission process eliminates one neutron and produces $\nu = 2.5$ new ones with a given probability q . The typical time over which this fission process occurs is $t_0 \approx 6\tau$ as we said that fission takes place one time over 6. Thus, the variation of the neutron number along the time dt is the product of the neutron number $n(t)$, the number of fissions during dt (dt/t_0), and the number of neutrons produced in each fission ($\nu q - 1$). The solution of Eq. (7.23) is

$$n(t) = n(0)e^{(\nu q - 1)\frac{t}{t_0}}. \quad (7.24)$$

This solution shows that the reaction can sustain itself if q is larger than $1/\nu = 0.4$.

In reality, this calculation should be changed because it is impossible to have a 100% enriched ^{235}U material. In such conditions one cannot achieve the above condition on q if one does not slow down neutrons so that the cross section increases. q is not only determined by the cross section value, but also by all the physical conditions that affect the neutron transport. Controlling a nuclear reactor means basically to control q . An uncontrolled increase of q may lead to catastrophic events due, for instance, to a lack of the control of the temperature.

7.A Appendix: relation between impact parameter and angular momentum

Let us start from the partial wave expansion of the scattering problem. If the solution of the radial Schrödinger equation with angular momentum l has the asymptotic form

$$u_l(r) \rightarrow e^{-i(kr - \frac{1}{2}l\pi)} - \eta_l e^{i(kr - \frac{1}{2}l\pi)}, \quad (7.25)$$

then one can find in standard quantum mechanics textbooks the proof that the elastic and reaction cross sections are given by

$$\begin{aligned} \sigma^{\text{el.}} &= \frac{\pi}{k^2} \sum_l (2l+1) |1 - \eta_l|^2, \\ \sigma^{\text{re.}} &= \frac{\pi}{k^2} \sum_l (2l+1) (1 - |\eta_l|^2). \end{aligned} \quad (7.26)$$

The total cross section is the sum

$$\sigma^{\text{tot}} = \frac{2\pi}{k^2} \sum_l (2l+1) (1 - \text{Re } \eta_l). \quad (7.27)$$

The largest possible reaction cross section is found when $\eta_l = 0$,

$$\sigma^{\text{re.}} = \frac{\pi}{k^2} \sum_l (2l+1) = \frac{2\pi}{k^2} \sum_l \left(l + \frac{1}{2} \right), \quad (7.28)$$

and if we wish to make a contact with the classical expression

$$\sigma^{\text{cl.}} = 2\pi \int db \, b, \quad (7.29)$$

then Eq. (7.7) must hold.