vertex, labeled on the right by the two entering line labels, and on the left by the two emerging line labels (whether particles or holes). Each propagating particle gives a time factor $\exp[i\epsilon_\rho(t_{i+1} - t_i)]$, and each hole, a factor $\exp[-i\epsilon_\lambda(t_{i+1} - t_i)]$.

It is in fact possible to proceed without explicitly denoting the intermediate states, instead drawing the diagrams first and deriving expressions for the contribution of each diagram to $U(t, t_0)$ from simple rules of association, as just described. There are some rules needed for specifying distinct diagrams, so that all contributions are counted properly. Rather than derive these directly, we shall turn to an alternative form of diagrammatic expansion, which does a very similar thing.

8.3 TIME ORDERING AND WICK'S CONTRACTION THEOREM

For the present development we shall use the Dyson form of the expansion of $U(t, t_0)$, given in Eq. 8. The essential differences between this expansion and that of Eq. 5 from which we generated the Goldstone expansion are the following: First, in Eq. 8 the interaction times $t_1, \ldots, t_n$ may be taken in any order, and do not appear in the limits of integration, while in Eq. 5 the times are ordered by the limits of integration. Second, the integrands in Eq. 8 contain the time-ordered products of interaction operators $V(t_i)$:

$$\{V(t_1) \cdots V(t_n)\}_T , \tag{22}$$

while Eq. 5 contains simple operator products. It is the time-ordered property we exploit now.

The basic method of evaluating an expression with time-ordered products of operators is that given by Wick, in which one rewrites the original expression in terms of another kind of product of the same operators, called a *normal-ordered* product, which will be defined shortly. We first illustrate the method with a simple example. Consider the time-ordered product of a fermion annihilation operator $a_\alpha(t)$ and a creation operator $a_\beta^\dagger(t')$, which may be written out as

$$\{a_\alpha(t)a_\beta^\dagger(t')\}_T = a_\alpha(t)a_\beta^\dagger(t')\theta(t-t') - a_\beta^\dagger(t')a_\alpha(t)\theta(t'-t) \tag{23}$$

by definition (see Eq. 7.24). Now we define the *normal-ordered* product (denoted by $\{\cdots\}_N$) of two such operators as follows:

$$\{a_\alpha(t)a_\beta^\dagger(t')\}_N = -a_\beta^\dagger(t')a_\alpha(t) ,$$
$$\{a_\beta^\dagger(t')a_\alpha(t)\}_N = a_\beta^\dagger(t')a_\alpha(t) . \tag{24}$$

The rule is that the given product is to be rewritten with the creation

operator to the left of the annihilation operator. The new product is assigned a positive sign if the new order can be reached from the given order by an even permutation of the fermion operators, and a negative sign if by an odd permutation. The times play no role in defining this product. For fermions we may also transform the hole operators for orbits below the Fermi sea, as in Eq. 7.18b. Then the normal-ordered product of b_λ and $b_\mu^\dagger(\lambda, \mu \le F)$ will be defined as $\pm b_\mu^\dagger b_\lambda$, as in Eq. 24 for particle operators $(\alpha, \beta > F)$. A normal-ordered product gives no contribution when operating between vacuum states; this leads to great simplification in expressions, as we see later.

We may relate the two ordered products of Eqs 23 and 24 by simple recombination of terms, to give

$$\{a_\alpha(t)a_\beta^\dagger(t')\}_T = \{a_\alpha(t)a_\beta^\dagger(t')\}_N + \{a_\alpha(t), a_\beta^\dagger(t')\}\theta(t-t') . \tag{25}$$

The last term contains the anticommutator of the a, $a^\dagger$ operators, which is easily evaluated using Eq. 7.11 for the interaction-picture operators:

$$\{a_\alpha(t), a_\beta^\dagger(t')\} = \delta_{\alpha\beta}e^{-i\epsilon_\alpha(t-t')} . \tag{26}$$

The last term of Eq. 25 is called the *contraction* of $a_\alpha(t)$ and $a_\beta^\dagger(t')$, that is, the remainder on replacing the T-product by the N-product, and is seen in Eq. 26 to be a c-number. Further, the contraction can be identified with the causal one-particle Green function for the noninteracting Fermi system given in Eq. 7.29, i.e. for $\alpha,\beta = \rho,\sigma > F$:

$$\{a_\rho(t)a_\sigma^\dagger(t')\}_T - \{a_\rho(t)a_\sigma^\dagger(t')\}_N = i\delta_{\rho\sigma}G_\rho^0(t-t') . \tag{27}$$

Similarly, the *contraction* of $b_\mu^\dagger(t)$ and $b_\lambda(t')$ may be easily expressed by the same Green function

$$\{b_\mu^\dagger(t)b_\lambda(t')\}_T - \{b_\mu^\dagger(t)b_\lambda(t')\}_N = -\delta_{\mu\lambda}e^{-i\epsilon_\lambda(t-t')}\theta(t'-t)$$
$$= i\delta_{\mu\lambda}G_\lambda(t-t') , \qquad \lambda \le F. \tag{28}$$

Note that ϵ_α and ϵ_λ are measured relative to the Fermi energy ϵ_F.

For completeness one may define the normal-ordered product for two fermion operators which anticommute as in Eq. 24, e.g.

$$\{a_\alpha(t)a_\beta(t')\}_N = a_\alpha(t)a_\beta(t') = -a_\beta(t')a_\alpha(t) , \tag{29}$$

where both orders on the right-hand side are normal-ordered, that is annihilators are to the right of creators. The *contraction* for such anticommuting operators vanishes:

$$\{a_\alpha(t)a_\beta(t')\}_T - \{a_\alpha(t)a_\beta(t')\}_N = \{a_\alpha(t), a_\beta(t')\}\theta(t-t') = 0\,, \quad (30)$$

since the anticommutator vanishes.

For any two fermion operators, we see from Eqs. 25 and 30 that the time-ordered product can be replaced by the normal-ordered product plus a contraction, which is always a c-number (and which may vanish). The advantage of the reordering becomes evident for ground-state perturbation theory, for which we shall evaluate expectation values of these expressions in the unperturbed ground state Φ_0. Then the normal-ordered products give vanishing contributions, since annihilators on the right or creators on the left will give zero acting on Φ_0. Thus, only the contractions will contribute: these can always be expressed in terms of the unperturbed Green function, as in Eqs. 27 and 28.

The generalization of this simple example to other operators proceeds as follows. Suppose we want to evaluate a time-ordered product of operators $\{ABC\cdots\}_T$ (where we suppress the time labels). First we decompose each operator $A, B, \ldots$ into products of its component creators $a^\dagger$, $b^\dagger$, and annihilators a, b. The entire product $(ABC\cdots)$ can then be written as a sum of products $(rstuv\cdots)$ of operators $r, s, t, \ldots$, each of which is one of the four: $a^\dagger$, $b^\dagger$, a, b. The time-ordered product $\{ABC\cdots\}_T$ can be expressed as a linear combination of time-ordered products of the form $\{rstuv\cdots\}_T$.

Now define the normal-ordered product $\{rstuv\cdots\}_N$ to be a reordering of $rstuv\cdots$ such that all creators stand to the left of all annihilators. The overall sign is given by the permutation required to obtain the new ordering from the original: $rstuv\cdots$. With this definition, one may now make use of a theorem due to Wick, which states that it is possible to rewrite the time-ordered products in the form

$$\begin{aligned}\{rstuv\cdots\}_T = \{&rstuv\cdots\}_N \\ &+ \{tuv\cdots\}_N\langle rs\rangle \\ &+ \{rsv\cdots\}_N\langle tu\rangle \pm \text{other pairs} \\ &+ \{v\cdots\}_N\langle rs\rangle\langle tu\rangle \pm \text{permutations} \\ &+ \cdots \\ &+ \langle rs\rangle\langle tu\rangle\langle vw\rangle\cdots \pm \text{permutations}\,, \end{aligned} \quad (31)$$

where $\langle rs\rangle$ stands for the pair contraction

$$\langle rs\rangle = \{rs\}_T - \{rs\}_N \quad (32)$$

given in Eqs. 25, 27, 28, and 30, with $\pm$ for even/odd permutations. The proof is inductive, and is given in the Appendix. (We have assumed an even number of operators in Eq. 31.)

The power of the result given in Eq. 31 is again clear if we want ground-state expectation values of time-ordered products, for which the normal-ordered products will not contribute. The result is that only the last line of Eq. 31 survives,

$$\langle\Phi_0|\{rstuv\cdots\}_T|\Phi_0\rangle = \langle rs\rangle\langle tu\rangle\langle vw\rangle\cdots \pm \text{permutations}, \quad (33)$$

giving an expression entirely in terms of pair contractions, which in turn can be written in terms of the unperturbed Green functions, as in Eqs. 27 and 28. We shall require just such expressions as Eq. 33 in the evaluation of the perturbation series of $U(t, t_0)$ (Eq. 8) for the ground state, with which we began this section. It is important for this result that we work in the interaction picture, in which Wick's contractions are c-numbers. We shall see how this works out in the next section.

8.4 FEYNMAN DIAGRAMS

The Wick method of handling time-ordering, which we have just studied, leads quite directly to the Feynman diagram representation of time evolution, and therefore also of perturbation theory. Let us see how this works by considering the expectation value of the time-development operator in the unperturbed ground state Φ_0, using the expansion given in Eq. 8,

$$\langle\Phi_0|U(t,t_0)|\Phi_0\rangle = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t} dt_n\,\langle\Phi_0|\{V(t_1)\cdots V(t_n)\}_T|\Phi_0\rangle\,. \quad (34)$$

For the present we shall study only the integrand of the nth term in Eq. 34, which is the expectation value of Eq. 22 in the state Φ_0.

To make use of the Wick analysis, we first write each of the $V(t_i)$ in terms of creators and annihilators at the same time t_i; i.e., for a two-body interaction,

$$V(t_i) = \frac{1}{2}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|v|\gamma\delta\rangle a_\alpha^\dagger(t_i)a_\beta^\dagger(t_i)a_\delta(t_i)a_\gamma(t_i)\,. \quad (35)$$

This operator is normal-ordered with respect to the physical vacuum, but not with respect to the Fermi ground state Φ_0. When we rewrite Eq. 35 making the particle–hole transformation of Eq. 7.16, some creators $b_\lambda^\dagger$ will appear to the right of annihilators a_δ or b_μ, and so on. Each term in nth order in Eq. 34 contains the expectation value of a time-ordered product of $4n$ operators $(a^\dagger b^\dagger ab)$ at n different times, to be integrated over the times as indicated. For each product, we use the Wick result of Eq. 33 to reduce the expression to a sum of products of Green functions of the various times, and

Appendix

A. WICK'S THEOREM: EQUATION 8.31

The decomposition of the time-ordered product $\{rstuv \cdots\}_T$ given by Eq. 8.31 is already proven for two operators, directly from Eq. 8.32, by interchanging terms:

$$\{rs\}_T = \{rs\}_N + \langle rs \rangle . \tag{A1}$$

Now we proceed inductively, first assuming the theorem to be true for a product of n operators $(rstuv \cdots z)$. (We have assumed n to be even.) Next we introduce one more operator, d, which we take to have a time *earlier* than all n operators $r, \ldots, z$, so that

$$\{rs \cdots z\}_T d = \{rs \cdots zd\}_T . \tag{A2}$$

On the other hand, for a normal-ordered product of m operators $w_1, \ldots, w_m$ (a subset of $r, \ldots, z$) we may obtain

$$\{w_1 \cdots w_m\}_N d = \{w_1 \cdots w_m d\}_N \pm \sum_{i=1}^{m} \{w_1 w_2 \cdots w_{i-1} w_{i+1} \cdots w_m\}_N \langle w_i d \rangle , \tag{A3}$$

where the factor w_i which appears in the contraction in the summation is missing from the normal-ordered product which multiplies it. This result is

clearly correct for the case that d is an annihilation operator, since then the first term is already normal-ordered, and the summation is then zero because every contraction $\langle w_i d \rangle = 0$. This last obtains because d occurs at a time earlier than all other operators: see Eqs. 8.25–8.30. If d is a creation operator, then Eq. A3 is obtained by commuting d through all the operators in the set $\{w_i\}$, obtaining one term in the sum from each commutation, and a $(\pm)$ sign for even or odd i.

Then, using Eq. 8.31 for the left side of Eq. A2 (by assumption), and applying Eq. A3 repeatedly, we find

$$\begin{aligned}\{rs \cdots zd\}_T = \{rs \cdots zd\}_N & \\ &+ \{tuv \cdots\}_N \langle rs \rangle + \{rsv \cdots\}_N \langle tu \rangle - \{s \cdots z\}_N \langle rd \rangle \\ &\pm \text{other pairs (from } r \cdots zd) \\ &+ \cdots + \\ &+ \langle rs \rangle \langle tu \rangle \cdots \langle yz \rangle d \pm \text{permutations} .\end{aligned} \tag{A4}$$

This is the Wick result for $n+1$ operators. For $n+2$ operators, we again obtain a result of the form of Eq. 8.31. That completes the proof.

B. LINKED-CLUSTER THEOREM: EQUATION 9.3

The theorem concerns the factoring of a given matrix element $\langle m|U(t, t_0)|n \rangle$ of the time-evolution operator between two states of the noninteracting system (m, n). This matrix element can be represented by an infinite series of Feynman diagrams, using the Wick analysis, as in Section 8.4. We have explained in Section 9.1 that diagrams may be grouped into those which are linked, and those which are not, meaning they contain disconnected parts, as in Figure 9.1. We discuss the theorem in terms of Figure 9.2, which shows the general form of a diagram with disconnected parts, in this case for an initial state with one particle and one hole, and a final state with two particles and two holes, which will serve as our examples of (m, n). The first property of interest for such a Feynman diagram is that the expression for $\langle m|U|n \rangle$ factors as

$$\langle m|U(t, t_0)|n \rangle_{\mathrm{a}} = \langle m|U(t, t_0)|n \rangle_{\mathrm{b}} \langle 0|U(t, t_0)|0 \rangle_{\mathrm{c}} , \tag{B1}$$

where a, b, c refer to the three parts of Figure 9.2. This is an example of Eq. 9.3, in which the first factor is the expression for the linked part of the diagram, and the second for the vacuum–vacuum part.

The factoring follows from the Wick method of constructing Feynman diagrams, as discussed in Section 8.4. Every particle or hole line connecting two interactions in the diagram corresponds to the Wick contraction of a pair of operators contained in those interactions. A diagram with two