Nuclear shapes

1 One way to parametrize the nuclear radius is

$$R(\vartheta,\varphi) = R_0 \left[1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y^*_{\lambda\mu}(\vartheta,\varphi) \right] \tag{1}$$

This expression is adopted by assuming that the parameters $\alpha_{\lambda\mu}$ are "small" in the sense that $|\alpha_{\lambda\mu}|^2 << |\alpha_{\lambda\mu}|$

2 It has to be noted that R_0 is not precisely the "radius". In fact if we assume that the variation of the shape with respect to sphericity does conserve volume (which also amounts to neglecting $(\lambda\mu) \equiv (0,0)$), then

$$\frac{4}{3}\pi R^3 = \int d\Omega \int_0^{R(\vartheta,\varphi)} dr\, r^2 = \int d\Omega \frac{R_0^3}{3}\left[1+\sum\right]^3$$

We neglect terms of order α^3 and use

$$\int d\Omega\, Y^*_{\lambda\mu} Y_{\lambda'\mu'} = \delta(\lambda\lambda')\,\delta(\mu\mu')$$

Then

$$\frac{4}{3}\pi R^3 = \int d\Omega \frac{R_0^3}{3}\left[1 + 3\sum_{\lambda\mu}\alpha_{\lambda\mu}Y^*_{\lambda\mu} + 3\sum_{\substack{\lambda\mu\\ \lambda'\mu'}}\alpha_{\lambda\mu}\alpha^*_{\lambda'\mu'} \times \right.$$

$$\left. \times\, Y^*_{\lambda\mu} Y_{\lambda'\mu'}\right] \underset{\alpha_{00}=0}{=} \int d\Omega \frac{R_0^3}{3}\left(1 + 3\sum_{\lambda\mu}|\alpha_{\lambda\mu}|^2\right) =$$

$$= \frac{4}{3}\pi R_0^3 \left(1 + 3\sum_{\lambda\mu}|\alpha_{\lambda\mu}|^2\right) \tag{2}$$

From this formula one can see the (small) difference between R_0 and R.

3 Not only $(\lambda\mu) \equiv 00$, but also $(\lambda\mu) \equiv (1\mu)$ are to be excluded from the sum.

At least for small α's, it can be shown that they are associated to a translation and not to a shape variation

Exercise: show that this is true for $(\lambda\mu) \equiv (10)$, knowing that

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta$$

4 Static deformations can be expected therefore for $\lambda \geq 2$.
Although some evidence has be found for octupole deformation $(\lambda = 3)$, in most of the known cases it is enough to restrict to quadrupole deformation, namely

$$R(\vartheta,\varphi) = R_0 \left[1 + \sum_{\mu=2}^{2} \alpha_{2\mu} Y^{*}_{2\mu}\right] \tag{3}$$

It is instructive to look at the same formula in cartesian coordinates

$$R = R_0 \left(1 + \alpha_{xx}\frac{x^2}{r^2} + \alpha_{yy}\frac{y^2}{r^2} + \alpha_{zz}\frac{z^2}{r^2} + 2\alpha_{xy}\frac{xy}{r^2} + \right.$$
$$\left. + 2\alpha_{xz}\frac{xz}{r^2} + 2\alpha_{yz}\frac{yz}{r^2}\right). \tag{4}$$

The relation between $\alpha_{2\mu}$ and α_{ij} can be found by knowing the expression of the spherical harmonics

$$Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\vartheta - 1\right) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\ \frac{2z^2 - x^2 - y^2}{r^2}$$

$$Y_{2\pm1} = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\vartheta\cos\vartheta\, e^{\pm i\varphi} =$$

$$= \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\vartheta\cos\vartheta\,(\cos\varphi \pm i\sin\varphi) = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\;\frac{xz \pm i\, yz}{r^2}$$

$$Y_{2,\pm2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\vartheta\, e^{\pm 2i\varphi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\vartheta\,(\cos\varphi \pm i\sin\varphi)^2$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\;\frac{x^2 - y^2 \pm 2i\,xy}{r^2}$$

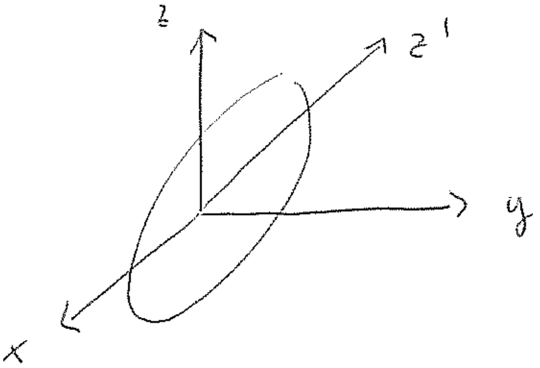
By inserting these expressions into (3), and comparing with (4)

$$\begin{cases} \alpha_{\pm2} = \sqrt{\frac{2\pi}{15}}\,(\alpha_{xx} - \alpha_{yy} \pm 2i\,\alpha_{xy}) \\ \alpha_{\pm1} = \mp 2\sqrt{\frac{2\pi}{15}}\,(\alpha_{xz} \pm i\,\alpha_{yz}) \\ \alpha_0 = \sqrt{\frac{\pi}{5}}\,\frac{2}{3}\,(2\alpha_{zz} - \alpha_{xx} - \alpha_{yy}) \end{cases} \tag{5}$$

5 The INTRINSIC frame is the one in which the α_{ij} tensor is diagonal. The axis of the intrinsic frame correspond to the principal symmetry axis. We call $O \equiv (x,y,z)$ the LAB system and $O' \equiv (x'y'z')$ the intrinsic system. There are three Euler angles that provide the transformation $O \to O'$. The Euler angles $(\vartheta, \psi, \varphi)$ are giving

- a rotation of ϑ around the original z-axis

- a rotation of φ around the new y-axis such that z goes to z'



- a rotation of ψ around z' such that $x \to x'$ and $y \to y'$

We denote these angles collectively by $\Omega \equiv \{\varphi, \vartheta, \psi\}$.

The spherical harmonics (and the α parameters) transform themselves according to

$$Y_{\lambda\mu}(\vartheta'\varphi') = \sum_{\mu'} D^{(\lambda)}_{\mu'\mu}(\Omega)\, Y_{\lambda\mu'}(\vartheta,\varphi) \qquad (6)$$

where the D are the Wigner D-functions.

In the intrinsic frame we call $a_{\lambda\mu}$ the $\alpha_{\lambda\mu}$ for the sake of clarity.

$$a_{\lambda\mu} \equiv \alpha_{\lambda\mu} \text{ (in the intrinsic frame) } = \sum_{\mu'} D^{(\lambda)}_{\mu'\mu}(\Omega)\, \alpha_{\lambda\mu'} \qquad (7)$$

We could use this formula but we prefer to derive the properties of the a's by recalling that the tensor a_{ij} in cartesian coordinates must be diagonal. From Eq (5)

$$a_{xy} = a_{yz} = a_{xz} = 0 \quad \Longrightarrow \quad a_{\pm1} = 0 \qquad a_{-2} = a_{+2} \qquad (8)$$

It looks quite natural that only two free parameters a_0 and a_2 remain, as the others have been replaced by the three

Euler angles. Usually one transforms (a_0, a_2) into (β, γ) through

$$\begin{cases} a_0 = \beta \cos\gamma \\ a_2 = \frac{1}{\sqrt{2}} \beta \sin\gamma \end{cases} \qquad (9)$$

Therefore,

$$R(\vartheta, \varphi) = R_0 \left(1 + a_0 Y_{20}^* + a_2 Y_{22}^* + a_2 Y_{2-2}^*\right).$$

Since $Y_{2-2} = Y_{22}^*$,

$$\frac{R - R_0}{R_0} = \beta \cos\gamma \, \frac{1}{4}\sqrt{\frac{5}{\pi}} \left(3\cos^2\vartheta - 1\right) + \frac{1}{\sqrt{2}} \beta \sin\gamma \times$$
$$\times \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\vartheta \times 2\cos(2\varphi)$$

$$R(\vartheta, \varphi) = R_0 \left\{ 1 + \sqrt{\frac{5}{16\pi}} \beta \left[\cos\gamma \left(3\cos^2\vartheta - 1\right) + \sqrt{3} \sin\gamma \times \right.\right.$$
$$\left.\left. \times \sin^2\vartheta \left(\cos^2\varphi - \sin^2\varphi\right) \right] \right\} \qquad (10)$$

This expression shows clearly that β is an overall size of the deformation.

To have a better insight, we calculate δR along the main axis

$$\delta R_x = R\left(\frac{\pi}{2}, 0\right) - R_0 = \sqrt{\frac{5}{16\pi}} \beta R_0 \left(-\cos\gamma + \sqrt{3}\sin\gamma \right)$$
$$= \sqrt{\frac{5}{4\pi}} \beta R_0 \left(\cos\gamma \left(-\frac{1}{2}\right) + \sin\gamma \, \frac{\sqrt{3}}{2} \right)$$

$$= \sqrt{\frac{5}{4\pi}}\, \beta R_0 \left(\cos\gamma \cos\frac{2\pi}{3} + \sin\gamma \sin\frac{2\pi}{3} \right)$$

Since $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha \sin\beta$

$$\boxed{\delta R_x = \sqrt{\frac{5}{4\pi}}\, \beta R_0 \cos\left(\gamma - \frac{2\pi}{3}\right)} \qquad (11)$$

$$\delta R_y = R\left(\frac{\pi}{2}, \frac{\pi}{2}\right) - R_0 = \sqrt{\frac{5}{16\pi}}\, \beta R_0 \left(-\cos\gamma - \sqrt{3}\sin\gamma \right)$$

$$= \sqrt{\frac{5}{4\pi}}\, \beta R_0 \left(\cos\gamma \left(-\frac{1}{2}\right) + \sin\gamma\left(-\frac{\sqrt{3}}{2}\right) \right)$$

$$= \sqrt{\frac{5}{4\pi}}\, \beta R_0 \left(\cos\gamma \cos\frac{2\pi}{3} - \sin\gamma \sin\frac{2\pi}{3} \right)$$

Since $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\boxed{\delta R_y = \sqrt{\frac{5}{4\pi}}\, \beta R_0 \cos\left(\gamma + \frac{2\pi}{3}\right)} \qquad (12)$$

$$\delta R_z = R(0,0) - R_0 = \sqrt{\frac{5}{16\pi}}\, \beta R_0\, 2\cos\gamma$$

$$\boxed{\delta R_z = \sqrt{\frac{5}{4\pi}}\, \beta R_0 \cos\gamma} \qquad (13)$$

6 • For $\gamma = 0$, $\delta R_x = \delta R_y < 0$ $\qquad \delta R_z > 0$

The shape has axial symmetry around the z axis and is elongated around that axis if we take $\beta > 0$. This shape is said to be prolate.

$$\delta R_x = \delta R_y = -\sqrt{\frac{5}{4\pi}}\, \frac{\beta}{2} R_0 \qquad \delta R_z = \sqrt{\frac{5}{4\pi}}\, \beta R_0$$

- If γ increases between 0 and $\frac{\pi}{3}$, the axial symmetry is lost (<u>triaxial shape</u>)

- If $\gamma = \frac{\pi}{3}$, then

$$\delta R_x = \sqrt{\frac{5}{4\pi}}\,\beta R_0 \cos\left(-\frac{\pi}{3}\right) = \sqrt{\frac{5}{4\pi}}\,\frac{\beta}{2} R_0$$

$$\delta R_y = -\sqrt{\frac{5}{4\pi}}\,\beta R_0$$

$$\delta R_z = \sqrt{\frac{5}{4\pi}}\,\frac{\beta}{2} R_0$$

The shape has again <u>axial symmetry</u> around an axis (in this case, y) but is elongated along the other two axis. Such a shape is called <u>oblate</u>.

- To each deformation we can associate a point in the 2D plane where $(r, \vartheta) = (\beta, \gamma)$. However, the points with $\beta > 0$ and $0 < \gamma < \frac{\pi}{3}$ are sufficient to describe every intrinsic shape