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Radiation to atom quantum mapping by collective recoil in a Bose–Einstein condensate

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Abstract

We propose a scheme to realize *radiation to atom* continuous variable quantum mapping, i.e., to teleport the quantum state of a single mode radiation field onto the collective state of atoms with a given momentum out of a Bose–Einstein condensate. The atoms-radiation entanglement needed for the teleportation protocol is established through the interaction of a single mode with the condensate in the presence of a strong far off-resonant pump laser, whereas the coherent atomic displacement is obtained by the same interaction with the radiation in a classical coherent field.
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Entanglement is a crucial resource in the manipulation of quantum information and quantum teleportation [1,2] is perhaps the most impressive example of quantum protocol based on entanglement. Teleportation is the transferral of (quantum) information between two distant parties that share entanglement. There is no physical move of the system from one player to the other and indeed the two parties need not even know each other's locations. Only classical information is actually exchanged between the parties. However, due to

entanglement, the quantum state of the system at the transmitter location (say Alice) is mapped onto a different physical system at the receiver location (say Bob). The information transferral is *blind*, i.e., the protocol should work also when the state to be teleported is completely unknown to both the sender and the receiver. Several teleportation protocols have been suggested either for qubits and continuous variable systems [3–11]. Moreover, interspecies teleportation schemes have been suggested either of atomic spin onto polarization states of light [12] or of motional state of a trapped ion and a light field [13].

In this letter, we propose a novel scheme to realize radiation to atom quantum state mapping,

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i.e., the interspecies teleportation of the quantum state of a single mode radiation field onto the collective state of atoms with a given momentum out of a Bose–Einstein condensate. The four basic ingredients of a quantum teleportation experiment are the following: (i) an entangled state shared between two parties; (ii) a joint *Bell measurement* performed on the system whose state is to be teleported and on one subsystems of the entangled state; (iii) a device able to perform a given class of unitary transformation, *conditioned* to the results of the joint measurement; (iv) a readout system to verify teleportation. In the following, we describe the above points for our teleportation protocol and discuss the feasibility conditions of our proposal. The setup is schematically illustrated in Fig. 1.
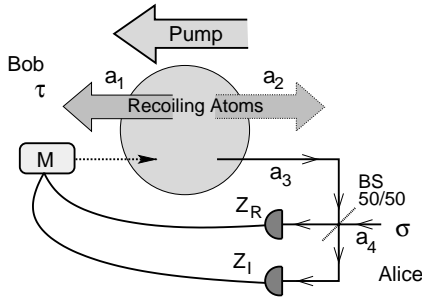

Fig. 1. Schematic diagram of the proposed setup to realize radiation to atom continuous variable quantum mapping, i.e., teleportation of the quantum state of a single mode radiation field onto the collective state of atoms with a given momentum out of a Bose–Einstein condensate. The protocol proceeds as follows: the atomic mode a_1 and the radiation mode a_3 are entangled through the interaction of the light mode with the condensate in presence of a strong far off-resonant pump laser (CARL dynamics). The outgoing radiation mode a_3 is then mixed (in a balanced beam splitter) at the sender' location (Alice) with another radiation mode a_4, excited in the state σ, which we want to teleport, and the joint measurement of a couple of two-mode quadratures is performed. The result of the measurement is sent to the receiver's location (Bob), where the corresponding coherent atomic displacement is performed. The latter is obtained through the same CARL interaction, by injecting a suitably modulated coherent pulse (M denotes a modulator). The overall dynamics is such that the ensemble of recoiling atoms in the mode a_1 is described by the density matrix τ, which approaches σ in the limit of high entanglement, i.e., high gain of the CARL interactions.

The entangled state supporting the teleportation protocol will be a *twin-beam-like* state of a radiation mode and a collective mode of atoms with a given momentum out of a Bose–Einstein condensate. This is obtained by the interaction of a Bose–Einstein condensate (BEC) with a single-mode quantized radiation field in the presence of a strong far off-resonant pump laser, in a regime well described by quantum collective atomic recoil laser (CARL) model [14]. The starting point of such a model is the classical Hamiltonian for N two-level atoms exposed to an off-resonant pump laser, whose electric field $\vec{E}_0 = \hat{e}\mathbf{E}_0\cos(\vec{k}_2 \cdot \vec{x} - \omega_2 t)$ is polarized along $\hat{e}$, propagates along the direction of $\vec{k}_2$ and has a frequency $\omega_2 = ck_2$ with a detuning from the atomic resonance, $\Delta_{20} = \omega_2 - \omega_0$, much larger than the natural linewidth of the atomic transition, γ. The atoms scatter a single-mode field circulating in a high-Q ring-cavity, with frequency ω_1, wave number $\vec{k}_1$ making an angle ϕ with $\vec{k}_2$ and electric field $\vec{E} = (\hat{e}/2)[\mathbf{E}(t)e^{i(\vec{k}_1 \cdot \vec{x} - \omega_1 t)} + \text{c.c.}]$ with the same polarization of the pump field. By adiabatically eliminating the internal atomic degree of freedom, the following CARL Hamiltonian can be derived [14]

$$H = \sum_{j=1}^{N}[\omega_r p_j^2 - ig(ae^{i\theta_j} - \text{c.c.})] - \Delta|a|^2,$$

where $\omega_r = \hbar q^2/2M$ is the recoil frequency, M is the atomic mass, $q = |\vec{q}|$ and $\vec{q} = \vec{k}_1 - \vec{k}_2$ is the difference between the scattered and the incident wave vectors, $\theta_j = qz_j$ and $p_j = p_{zj}/\hbar q$ are the dimensionless position and momentum of the jth atom along the axis z directed along $\vec{q}$, $g = (\Omega_0/2\Delta_{20})(\omega_2 d^2/2\hbar\epsilon_0 V)^{1/2}$, $a = -i(\epsilon_0 V/2\hbar\omega_2)^{1/2}\mathbf{E}e^{i\Delta t}$, $\Delta = \omega_2 - \omega_1$, $\Omega_0 = d\mathbf{E}_0/\hbar$ is the Rabi frequency of the pump, V is the interaction volume, d is the atomic dipole and ϵ_0 is the permittivity of the free space.

In order to quantize both the radiation field and the center-of-mass motion of the atoms, θ_j, p_j and a are considered as quantum operators satisfying the canonical commutation relations $[\theta_j, p_{j'}] = i\delta_{j,j'}$ and $[a, a^\dagger] = 1$. The model is then second quantized introducing the atomic field operator $\Psi(\theta)$ with equal-time commutation relations $[\Psi(\theta), \Psi^\dagger(\theta')] = \delta(\theta - \theta')$, $[\Psi(\theta), \Psi(\theta')] = 0$ and the nor-

malization condition $\int_0^{2\pi} d\theta\, \Psi(\theta)^\dagger \Psi(\theta) = N$. Creation and annihilation operators are introduced for atoms with a definite momentum p, i.e., $\Psi(\theta) = \sum_m c_m u_m(\theta)$, where $u_m(\theta) = \exp(im\theta)/\sqrt{2\pi}$ and c_m are bosonic operators obeying the commutation relations $[c_m, c_n^\dagger] = \delta_{m,n}$ and $[c_m, c_n] = 0$, and $c_m^\dagger$ creates an atom with momentum $p = m$ in $\hbar q$ unit. This description of the atomic motion in a BEC assumes that the atoms are delocalized inside the condensate and that, at zero temperature, the momentum uncertainty $\sigma_{p_z} \approx \hbar/\sigma_z$ can be neglected with respect to $\hbar q$. The approximation is valid for $L \gg \lambda$, if $\sigma_z \approx L$, where L is the size of the condensate. The Hamiltonian for the second quantized model becomes [15]

$$H = \sum_{n=-\infty}^{\infty} \left\{ \omega_{\mathrm{r}} n^2 c_n^\dagger c_n + \mathrm{i}g\left(a^\dagger c_n^\dagger c_{n+1} - \mathrm{h.c.}\right) \right\} - \Delta a^\dagger a.$$

(1)

Notice that the Hamiltonian (1) commutes with the number of atoms, $N = \sum_n c_n^\dagger c_n$, and the total momentum, $P = a^\dagger a + \sum_n n c_n^\dagger c_n$. Let us now consider the equilibrium state with no field and all the atoms at rest, i.e., in momentum state with $n = 0$. In the linear regime, we neglect the atomic depletion of the ground state $n = 0$, taking $c_0 \approx \sqrt{N}$ as a c-number, and we consider only the transitions induced by the radiation field from the state $n = 0$ toward the levels $n = -1$ and $n = 1$. Introducing the operators $a_1 = c_{-1} \exp(\mathrm{i}\Delta t)$, $a_2 = c_1 \exp(-\mathrm{i}\Delta t)$ and $a_3 = a \exp(-\mathrm{i}\Delta t)$, the atomic field operator reduces to

$$\Psi(\theta, t) \approx \frac{1}{\sqrt{2\pi}} \left\{ \sqrt{N} + a_1(t)\mathrm{e}^{-\mathrm{i}(\theta + \Delta t)} + a_2(t)\mathrm{e}^{\mathrm{i}(\theta + \Delta t)} \right\}$$

(2)

and the Hamiltonian (1) reduces to the effective Hamiltonian [16]

$$H_0 = \delta_+ a_2^\dagger a_2 - \delta_- a_1^\dagger a_1 + \mathrm{i}g\sqrt{N}[(a_1^\dagger + a_2)a_3^\dagger - \mathrm{h.c.}],$$

(3)

where $\delta_\pm = \Delta \pm \omega_{\mathrm{r}}$. Hence, the dynamics of the system is that of three parametrically coupled harmonic oscillators a_1, a_2 and a_3, which obey the commutation rules $[a_i, a_j] = 0$ and $[a_i, a_j^\dagger] = \delta_{i,j}$ for $i, j = 1, 2, 3$. Notice that the Hamiltonian (3) admits $C = a_2^\dagger a_2 - a_1^\dagger a_1 + a_3^\dagger a_3$ as a constant of motion.

Hence, in the linear regime, the quantum CARL Hamiltonian reduces to that for three coupled modes, the first two modes a_1 and a_2 corresponding to atoms having lost or gained, respectively, a two-photon recoil momentum $\hbar q$, and the third mode a_3 corresponding to the photons of the scattered field. Starting from the vacuum $|\mathbf{0}\rangle = |0\rangle_1 |0\rangle_2 |0\rangle_3$ the state at a given time is given by [16]

$$|\Psi\rangle = \frac{1}{\sqrt{1 + N_1}} \sum_{m,n=0}^{\infty} \alpha^m \beta^n \sqrt{\frac{(m+n)!}{m!n!}} |m+n, m, n\rangle,$$

(4)

where $|\alpha|^2 = (N_2)/(1 + N_1)$, $|\beta|^2 = (N_3)/(1 + N_1)$, and N_1, N_2 and N_3 are the (time-dependent) average numbers of quanta of the three oscillators [see Eqs. (6)–(8)]. Since we start from vacuum we have $N_1 = N_2 + N_3$ at any time.

Eq. (4) shows that, in general, the system is entangled and that the distribution over the different occupation numbers, N_1, N_2 and N_3, is thermal. In particular, for $N_3 \ll N_1 \sim N_2$, the state $|\Psi\rangle$ reduces to

$$|\psi_{12}\rangle = \frac{1}{\sqrt{1 + N_1}} \sum_{n=0}^{\infty} \alpha^n |n, n, 0\rangle,$$

showing maximal entanglement between atoms with different momenta. On the other hand, for $N_2 \ll N_1 \approx N_3$, $|\Psi\rangle$ reduces to

$$|\psi_{13}\rangle = \frac{1}{\sqrt{1 + N_1}} \sum_{n=0}^{\infty} \beta^n |n, 0, n\rangle,$$

(5)

showing maximal entanglement between atoms and photons. Both the states $|\psi_{12}\rangle$ and $|\psi_{13}\rangle$ are pure bipartite states. They are maximally entangled states for the given number of quanta, according to the excess von Neumann entropy criterion [17], whereas the presence of a third mode reduces, in general, the entanglement between the other two modes [18]. The atom-radiation entangled state (5) is what supports our teleportation scheme. Incidentally, it has the same form of the twin-beam state of radiation used to realize continuous variable optical teleportation [9], and this will allows us to employ the same kind of Bell measurement scheme.

In the quantum limit $g\sqrt{N} \ll \omega_r$ and for $g\sqrt{N}t \gg 1$ the population of the three oscillators are given by

$$N_1(t) \approx \frac{1}{4}\left[1 + \left(\frac{g\sqrt{N}}{2\omega_r}\right)^2\right]e^{2g\sqrt{N}t}, \tag{6}$$

$$N_2(t) \approx \frac{1}{4}\left(\frac{g\sqrt{N}}{2\omega_r}\right)^2 e^{2g\sqrt{N}t}, \tag{7}$$

$$N_3(t) \approx \frac{1}{4}e^{2g\sqrt{N}t} \tag{8}$$

so that $N_1 \approx N_3 \gg N_2$. Furthermore, maximal entanglement between modes 1 and 3 requires $N_2 \leqslant 1$, so that the interaction time must satisfy the following limits:

$$\frac{1}{g\sqrt{N}} \ll t_{int} \leqslant \frac{1}{g\sqrt{N}} \ln\left(\frac{4\omega_r}{g\sqrt{N}}\right). \tag{9}$$

The state σ we want to teleport onto the atomic mode a_1 pertains to an additional radiation mode a_4. The Bell measurement is jointly performed on a_3 and a_4, and consists in the measurement of the real and the imaginary part of the complex operator $Z = a_3 + a_4^\dagger$. The measurement of $Z_R = \text{Re}[Z]$ and $Z_I = \text{Im}[Z]$ corresponds to measuring the sum- and difference-quadratures $x_3 + x_4$ and $y_3 - y_4$ of the two modes, where the quadrature x of a mode b is the operator $(b + b^\dagger)/2$, and the quadrature y is the operator $(b - b^\dagger)/2i$. Such a measurement can be experimentally implemented by multiport homodyne detection (i.e., by mixing the two modes in balanced beam splitter and then measuring two conjugated quadratures on the outgoing modes, see Fig. 1), if the two modes have the same frequencies [19,20], or by heterodyne detection otherwise [21]. The measurement is described by the following probability operator-valued measurement (POVM) [22], acting on the Hilbert space of mode a_3

$$\Pi_\alpha = \frac{1}{\pi}D(\alpha)\sigma^T D^\dagger(\alpha), \tag{10}$$

where α is a complex number, $D(\alpha)$ is the displacement operator $D(\alpha) = \exp\{\alpha a_3^\dagger - \bar{\alpha}a_3\}$ and $(\cdot)^T$ denotes the transposition operation. The result of the measurement is classically transmitted to the receiver's location (Bob), where a displacement

operation $D(\alpha)^\dagger$ is performed on the conditional state ϱ_α (see below on how to implement coherent atomic displacement). The dynamics of the conditional measurement is described by [22]

$$p_\alpha = \text{Tr}_{13}[|\psi_{13}\rangle\langle\psi_{13}|\mathbf{I}_1 \otimes \Pi_\alpha],$$
$$\varrho_\alpha = \frac{1}{p_\alpha}\text{Tr}_3[|\psi_{13}\rangle\langle\psi_{13}|\mathbf{I}_1 \otimes \Pi_\alpha], \tag{11}$$
$$\tau_\alpha = D(\alpha)\varrho_\alpha D^\dagger(\alpha),$$

where p_α is the probability for the result α in the Bell measurement, $\mathbf{I}$ is the identity operator, ϱ_α is the conditioned state of the atomic beam after the measurement, and τ_α is the conditioned state after the displacement operation. In other words, τ_α describes the conditioned state (sub-ensemble) which would have been selected if only the runs with outcome α from the heterodyne measurement were accepted. The teleported state (global ensemble) corresponds to the average over all the possible outcomes, i.e.,

$$\tau = \int_{\mathbb{C}} d^2\alpha\, p_\alpha\tau_\alpha$$
$$= \int_{\mathbb{C}} d^2\alpha\, D(\alpha)\text{Tr}_3[|\psi_{13}\rangle\langle\psi_{13}|\mathbf{I}_1 \otimes \Pi_\alpha]D^\dagger(\alpha). \tag{12}$$

After performing the partial trace and with some algebra [22], one has

$$\tau = \int \frac{d^2\alpha}{\pi K}\exp\left\{-\frac{|\alpha|^2}{K}\right\}D(\alpha)\sigma D^\dagger(\alpha), \tag{13}$$

where

$$K = 1 + N_1 + N_3 - \sqrt{(N_1 + N_3)(N_1 + N_3 + 2)}.$$

Eq. (13) shows that the overall dynamics of our scheme is equivalent to that of a Gaussian noisy channel with parameter K [22,23]. The density matrix τ, describing the final state of the atomic mode a_1 coincides with σ in the limit $N_1 + N_3 \to \infty$, i.e., for high gain in the CARL dynamics. Notice that N, and in turn N_1 and N_3, may vary in the repeated preparations of the condensate, thus introducing additional noise in the teleported state.

The displacement operation $D(\alpha)$ that should be performed on the conditional atomic state ϱ_α can be obtained using the same CARL Hamiltonian in

the condensate, by injecting a suitably modulated pulse, i.e., by exciting the mode a_3 in a classical coherent state. In this case, assuming a short pulse, the effective Hamiltonian may be written as $H_2 = ig\sqrt{N}\gamma(a_1 + a_2 + \text{h.c.})$, where γ is the amplitude of the modulated pulse. The terms proportional to $a_j^\dagger a_j$, $j = 1, 2$ in (3) can be discarded, and the evolution operator $U = \exp(iH_2\tau) = D_1\,(\alpha)\otimes D_2^\dagger(\alpha)$ coincides with the product of two displacement operators, one for each of the atomic modes, with amplitude given by $\alpha = -g\bar{\gamma}\sqrt{N}\tau$, where τ is an effective interaction time. The amplitude γ of the pulse can suitably tuned to obtain the desired value of the amplitude α, matching the results of the Bell measurements. The above dynamics displaces both the atomic modes, however without introducing quantum correlations. Therefore, we just ignore the effect on the atomic mode a_2, which does not participate to the teleportation protocol.

The time duration of the pulse should be small when compared to the time scale of the CARL dynamics and the decoherence time of $|\psi_{13}\rangle$ under free evolution. This is in order for two reasons: on the one hand we have that the CARL dynamics should be switched off after producing the desired entangled state $|\psi_{13}\rangle$, and therefore the whole protocol should be completed within the decoherence time. On the other hand, the displacement should be performed on a time scale comparable to that of the Bell measurement, i.e., before the *reset* of the dynamics and the generation of the subsequent copy of the atom-radiation entangled state in the new condensate by CARL. Overall, our protocol may be described as a feedforward control scheme, randomized according to the statistics of the Bell measurement.

In order to discuss the readout part of our scheme, we should go back to the initial entangled scheme produced by the CARL dynamics. This should be more properly written as $|\psi_{13}\rangle = (1 + N_1)^{-1/2} \sum_{n=0}^N \beta^n |n, 0, n, N - n\rangle$, where the fourth entry in the ket describes the number of atoms in the condensate. Since N is a large number (of the order of 10^5–10^6) writing the state as in Eq. (5) is perfectly admissible as far as we are concerned with its dynamics. However, this should be

taken into account if we want to reconstruct the state of the output atomic beam. Let us consider, for the sake of simplicity, the initial light signal state as a pure state $\sigma = |\varphi\rangle\langle\varphi|$, $|\varphi\rangle = \sum_n \varphi_n |n\rangle$. In the limit of high CARL gain the teleported state on the atomic beam is given by $|\varphi'\rangle = \sum_n \varphi_n |n, N - n\rangle$. This indicates that any proper verification of the teleportation should involve a measurement also on the condensate, e.g., a two mode tomographic method involving both the measurement of both momentum-mode and condensate quadratures [24,25]. Such kind of measurements are at present experimentally challenging and therefore, in order to obtain an accessible readout system, we propose to check only the statistics of the population $|\varphi_n|^2$, i.e., the diagonal part of the teleported state, which can be achieved by counting atoms. By choosing the initial radiation state $|\varphi\rangle$ as a squeezed vacuum or a Fock number state, we obtain an atomic teleported state with an even–odd or sub-Poissonian atomic number distribution. Although this kind of measurements would be only a partial verification of teleportation it would show the transferral of a nonclassical such as sub-Poissonian statistics. In turn, this implies that nonlocal correlations between the input radiation mode and the output atomic mode has been established and exploited.

In conclusion, we have proposed a novel scheme to realize the interspecies teleportation of the quantum state of a single mode radiation field onto the collective state of atoms with a given momentum out of a Bose–Einstein condensate. The entangled resource needed for the teleportation protocol is established through the CARL interaction of a single mode with the condensate in presence of a strong far off-resonant pump laser, whereas the coherent atomic displacement is obtained through the same interaction by injecting a suitably modulated short coherent pulse. The present description assumed an ideal BE condensate where decoherence effects have been neglected. Indeed, the main goal of the present paper has been to put forward the idea of interspecies teleportation in coherent atomic systems rather then discussing the details of possible implementations. A more realistic, and also unavoidably more technical treatment, is in progress, also in

view of an experimental realization, and results will be reported elsewhere.

References

[1] C.H. Bennett et al., Phys. Rev. Lett. 70 (1993) 1895.
[2] S.L. Braunstein et al., Phys. Rev. A 64 (2001) 022321;
R. van Loock, S.L. Braunstein, Phys. Rev. Lett. 87 (2001) 247901;
S.L. Braunstein, H.J. Kimble, Phys. Rev. A 61 (2000) 042302;
S.L. Braunstein et al., J. Mod. Opt. 47 (2000) 267.
[3] D. Bouwmeester et al., Nature 390 (1997) 575.
[4] D. Boschi et al., Phys. Rev. Lett. 80 (1998) 1121.
[5] M. Nielsen et al., Nature 396 (1998) 52.
[6] Y. Kim et al., Phys. Rev. Lett. 86 (2001) 1370.
[7] E. Lombardi et al., Phys. Rev. Lett. 88 (2002) 070402.
[8] I. Marcikic et al., Nature 421 (2003) 509.
[9] A. Furusawa et al., Science 282 (1998) 706.
[10] S. Babichev, et al. Available from <quant-ph/0208066>.
[11] W. Bowen, et al. Available from <quant-ph/0207179>.
[12] A. Kuzmich, E.S. Polzik, Phys. Rev. Lett. 85 (2000) 5639;
C. Schori et al., Phys. Rev. Lett. 89 (2002) 057903;
J. Hald, E.S. Polzik, J. Opt. B 3 (2001) S83.
[13] A.S. Parkins, H.J. Kimble, J. Opt. B 1 (1999) 496.
[14] R. Bonifacio, L. De Salvo Souza, Nucl. Instrum. Meth. Phys. Res. A 341 (1994) 360;
R. Bonifacio, L. De Salvo Souza, L.M. Narducci, E.J. D'Angelo, Phys. Rev. A 50 (1994) 1716;
R. Bonifacio, L. De Salvo, Appl. Phys. B 60 (1995) S233.
[15] N. Piovella, M. Cola, R. Bonifacio, Phys. Rev. A 67, 013817.
[16] M.G. Moore, O. Zobay, P. Meystre, Phys. Rev. A 60 (1999) 1491.
[17] G. Lindblad, Comm. Math. Phys. 33 (1973) 305;
S.M. Barnett, S.J.D. Phoenix, Phys. Rev. A 44 (1991) 535.
[18] T. Gasenzer, D.C. Roberts, K. Burnett, Phys. Rev. A 65 (2002) 021605(R).
[19] G.M. D'Ariano, M.F. Sacchi, Phys. Rev. A 52 (1995) R4309.
[20] M.G.A. Paris, A. Chizhov, O. Steuernagel, Opt. Commun. 134 (1997) 117.
[21] H. Yuen, J. Shapiro, IEEE Trans. Inf. Theory IT-26 (1980) 78.
[22] M.G.A. Paris, M. Cola, R. Bonifacio, Phys. Rev. A 67 (2003) 042104;
J. Opt. B 5 (2003) S360.
[23] H.F. Hofmann et al., Phys. Rev. A 62 (2000) 062304;
Masashi Ban, et al. Available from preprint <quant-ph/0202172>.
[24] For a recent review see: G.M. D'Ariano, M.G.A. Paris, M.F. Sacchi, quant-ph/0302028, in: P. Hawkes (Ed.), Advances in Imaging and Electron Physics, to appear.
[25] J.F. Corney, G.J. Milburn, Phys. Rev. A 58 (1998) 2399.