# Quantum Free Electron Laser From 1D to 3D

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# OUTLOOK

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- REVIEW OF 1D QFEL MODEL

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- WORKING EQUATIONS

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- NEW 3D MODEL FOR QFEL
- WORKING EQUATIONS
- CONCLUSION

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- Propagation effects by Bonifacio, Piovella, Robb, Cola, Opt. Commun. 252 (2005)
  Multiple scaling approach

# FEL PARAMETER

- $\lambda_r = \frac{1-\beta_{\|}}{\beta_{\|}}\lambda_w \simeq \lambda_w \frac{1+a_w^2}{2\gamma^2} \quad \lambda_l = 2\lambda_w$ Resonant Wavelength
- $\gamma_r = \sqrt{\frac{\lambda_w(1+a_w^2)}{2\lambda}}$ Resonant energy
- $p = \frac{mc(\gamma-\gamma_0)}{\hbar k}$ Momentum in $\hbar k$ unit
- $\theta = (k + k_w)z - c(k - k_w)t$ Phase
- $\rho = \frac{1}{\gamma_r}\left(\frac{a_w \omega_p}{4 k_w c}\right)^{2/3}$ Classical Fel parameter
- $\bar{\rho} = \rho\frac{mc\gamma_r}{\hbar k} = \gamma_r \rho \frac{\lambda_r}{\lambda_c}$ Quantum Fel parameter
- $\bar{z} = z/L_g \quad \bar{p} = p/\bar{\rho}$ Rescaled Position and Momentum
- $\delta = \frac{\gamma_0 - \gamma_r}{\rho\gamma_r} \quad L_g = \lambda_w/4\pi\rho$ Detuning and Gain length

# 1D QFEL MODEL I

*Hamiltonian and particle equations*

$$H^I(\theta,p) = \sum_{j=1}^{N} H_j^I(\theta,p) \qquad [\theta_j, p_k] = i\delta_{jk} \quad \left[a, a^\dagger\right] = 1$$

$$H_j^I(\theta,p) = \left\{ \frac{p_j^2}{2\bar{\rho}} - i\sqrt{\frac{\bar{\rho}}{N_e}}\left(ae^{i\theta_j} - h.c.\right) - \delta a^\dagger a \right\}$$

$$\partial_{\bar{z}}\theta_j = \frac{p_j}{\bar{\rho}} = \partial_{\bar{p}_j} H$$

$$\partial_{\bar{z}} p_j = -\sqrt{\frac{\bar{\rho}}{N_e}}\left(ae^{i\theta_j} + h.c.\right) = -\partial_{\bar{\theta}_j} H$$

$$d_{\bar{z}} a = \sqrt{\frac{\bar{\rho}}{N_e}} \sum_{j=1}^{N} e^{-i\theta_j} + i\delta a$$

# 1D QFEL MODEL II

N particle described by matter-wave field $\hat{\psi}(\theta, \bar{z})$

$$\hat{\psi}(\theta, \bar{z}) = \sum_{n \in Z} \hat{c}_n(\bar{z}) e^{i\theta n}, \qquad \int_0^{2\pi} d\theta \, \hat{\psi}^\dagger(\theta)\hat{\psi}(\theta) = \hat{N}$$

The electrons are treated like BOSONS !!!

$$\left[\hat{c}_n, \hat{c}^\dagger_{n'}\right] = \delta_{nn'} \Longleftrightarrow \left[\hat{\psi}(\theta), \hat{\psi}^\dagger(\theta')\right] = \delta(\theta - \theta')$$

$$\hat{H}^{II} = \int_0^{2\pi} d\theta \, \hat{\psi}^\dagger(\theta) \, H^I(\theta, -i\partial_\theta, a, a^\dagger) \, \hat{\psi}(\theta)$$

N. Piovella, M.M. Cola and R. Bonifacio PRA 67. (2003)

# 1D QFEL MODEL III

$$
\begin{aligned}
i\partial_{\bar z}\hat\psi &= -\frac{1}{2\bar\rho}\partial^2_{\bar\theta}\hat\psi - i\sqrt{\frac{\bar\rho}{N_e}}\left(ae^{i\theta} - c.c.\right)\hat\psi \\
d_{\bar z}a &= \sqrt{\frac{\bar\rho}{N_e}}\int_0^{2\pi} d\theta\,\hat\psi^\dagger(\theta)\,e^{-i\theta}\hat\psi(\theta) + i\delta a
\end{aligned}
$$

- Preparata Hypotesis $(N_e \to \infty)$ $\hat\psi \to \sqrt{N_e}\psi$ and $\hat a \to \sqrt{\bar\rho N_e}A$
- introducing the time-dependence in field $A(\bar z) \to A(\bar z, \theta)$

$$
i\partial_{\bar z}\psi(\theta,\bar z) = -\frac{1}{2\bar\rho}\partial^2_{\bar\theta}\psi(\theta,\bar z) - i\bar\rho\left(A(\bar z,\theta)e^{i\theta} - c.c.\right)\psi(\theta,\bar z)
$$

$$
\partial_{\bar z}A(\bar z,\theta) + \frac{1}{2\rho}\partial_\theta A(\bar z,\theta) = \left|\psi(\theta,\bar z)\right|^2 e^{-i\theta} + i\delta A(\bar z,\theta)
$$

# MULTIPLE SCALING APPROACH

Including propagation effect we introduce a slow scale:

$$z_1 = \epsilon\theta = \frac{z - v_r t}{\beta_r L_c} \quad \epsilon = 2\rho \quad L_c = \frac{\lambda}{4\pi\rho} \quad \beta_r = v_r = \frac{ck}{k + k_w}$$

$$\frac{1}{\epsilon}\frac{\partial}{\partial\theta} \rightarrow \frac{1}{\epsilon}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z_1}$$

*performing a perturbation expansion to the first order in $\epsilon$ we obtain:*

$$i\partial_{\bar{z}}\psi(\theta, \bar{z}, z_1) = -\frac{1}{2\bar{\rho}}\partial^2_{\bar{\theta}}\psi(\theta, \bar{z}, z_1) - i\bar{\rho}\left(A(\bar{z}, z_1)e^{i\theta} - c.c.\right)\psi(\theta, \bar{z}, z_1)$$

$$\partial_{\bar{z}}A(\bar{z}, z_1) + \partial_{z_1}A(\bar{z}, z_1) = \int_0^{2\pi} d\theta\ |\psi(\theta, \bar{z}, z_1)|^2\ e^{-i\theta} + i\delta A(\bar{z}, z_1)$$

R. Bonifacio, N.Piovella, G.R.M. Robb, M.M. Cola, Opt. Commun. 252 (2005)

# 1D WORKING EQUATIONS

$$\psi(\theta, \bar{z}, z_1) = \frac{1}{\sqrt{2\pi}} \sum_{n \in Z} c_n(\bar{z}, z_1) e^{in(\theta + \delta \bar{z})} \qquad \int_0^{2\pi} d\theta \, |\psi(\theta, \bar{z}, z_1)|^2 = I(z_1)$$

$$\partial_{\bar{z}} c_n = -i \left( \frac{n^2}{2\bar{\rho}} + n\delta \right) c_n - \bar{\rho} \left\{ A c_{n-1} - A^* c_{n+1} \right\}$$

$$\partial_{\bar{z}} A + \partial_{z_1} A = \sum_{n \in Z} c_n c^*_{n-1}$$

$|c_n(\bar{z}, z_1)|^2 =$ Probability to find an electron whit momentum $p = n(\hbar k)$ at z and $z_1$

$\frac{1}{N_e} \sum_{n \in Z} c_n c^*_{n-1} =$ Bunching operator

# THE ENERGY SPREAD

We must include the more physical consistent situation of an initial distribution for the electron energy.
Infact each electron of the beam has different initial energy.

$$\delta = \frac{\gamma - \gamma_0}{\rho \gamma_r} \rightarrow \delta_i = \frac{\gamma - \gamma_{0_i}}{\rho \gamma_r}$$

$$c_n(\bar{z}, z_1) \rightarrow c_n(\delta, \bar{z}, z_1)$$

$$\sum_{n \in Z} c_n c_{n-1}^* \rightarrow \int_R d\delta \, G(\delta) \sum_{n \in Z} c_n(\delta) c_{n-1}^*(\delta)$$

$G(\delta)$ is normalized distribution center around $\delta = 1/2\bar{\rho}$

# Self Aplified Spontaneus Emission SASE

Ingredient of SASE:

- Starting from noise
- Propagation effects
- Superradiant instability (the electrons radiate as $N_e^2$)
  Each cooperation length in the e-beam radiates a SR spike which is amplified when it propagates forward on the beam

SASE in the quantum regime:

- In the quantum regime the FEL behaves like a two level system
- Electrons emit coherent photons as in a LASER
- in the quantum SASE mode the spectrum is intrinsically narrow (quantum purification)

# CLASSICAL AND QUANTUM REGIME

$$\boxed{\bar{\rho} = \rho m c \gamma_r / \hbar k}$$

- If $\bar{\rho} \gg 1$ ,since $\frac{\delta\gamma}{\gamma} \simeq \rho \rightarrow mc(\delta\gamma) \gg \hbar k \rightarrow$ Classical behavior
  Many recoils implies many photons, hence classically, each electron emitS many photons

$$\bar{\rho}|A|^2 \simeq \langle N \rangle_{ph} / N_e \gg 1$$

- If $\bar{\rho} \lesssim 1 \rightarrow mc(\delta\gamma) \lesssim \hbar k \rightarrow$ Quantum behavior
  Each electron emits only a single photon therefore quantum FEL behaves like a two-level system

$$\bar{\rho}|A|^2 \simeq \langle N \rangle_{ph} / N_e \simeq 1$$

# QUANTUM 1D LINEAR THEORY I

Performing a linear analysis and looking for solutions proportional to $e^{i(\lambda\bar{z}+\bar{\omega}z_1)}$ we obtain the dispersion relation:

$$\left(\lambda-\Delta_n\right)\left(\lambda^2-\frac{1}{4\bar{\rho}^2}\right)+1=0$$

$$\Delta_n=\delta+\frac{n}{\bar{\rho}}-\bar{\omega},$$

Note that:

For $n=0$ and $\bar{\omega}=0 \rightarrow (\lambda-\delta)\left(\lambda^2-\frac{1}{4\bar{\rho}^2}\right)+1=0$

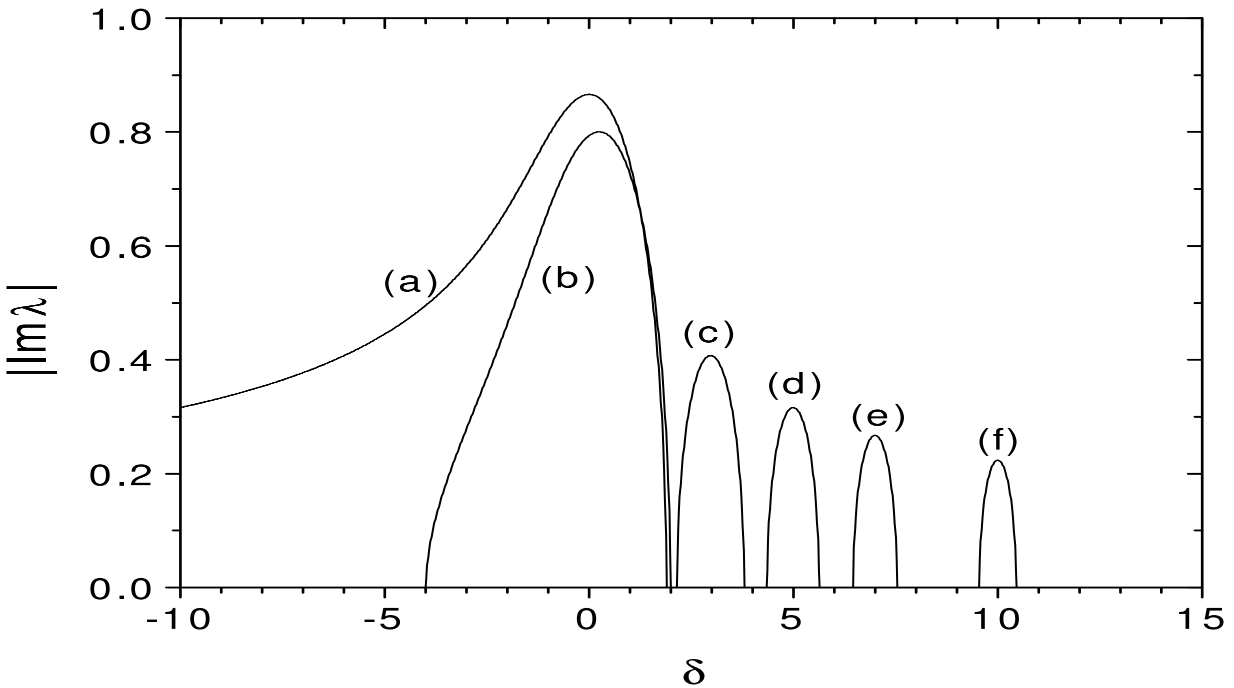
Quantum steady state dispersion relation

And for $\bar{\rho}\gg 1 \rightarrow (\lambda-\delta)\,\lambda^2+1=0$

Classical steady state dispersion relation

R. Bonifacio, N. Piovella, G.R.M. Robb and A. Schiavi PRST 9, (2006)

# QUANTUM 1D LINEAR THEORY II



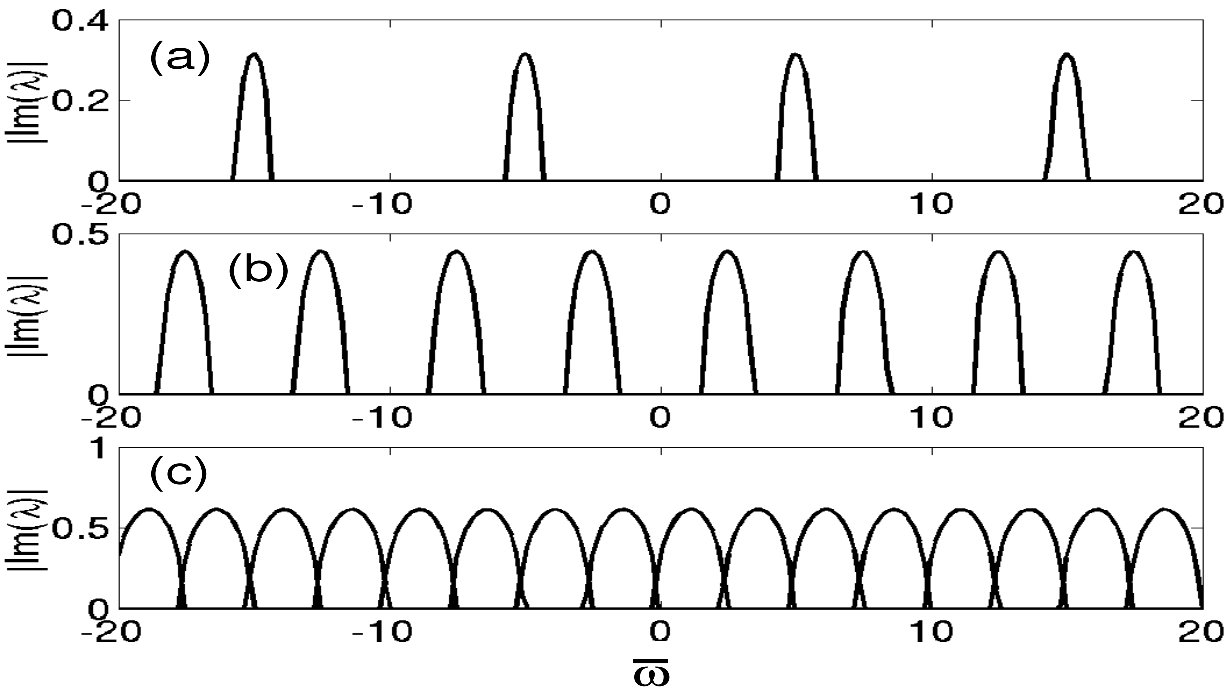

when $\bar{\rho} \lesssim 1 \to \delta = \frac{1}{2\bar{\rho}}$, width=$4\sqrt{\bar{\rho}}$ Peak of $\Im\lambda = \sqrt{\bar{\rho}}$

$\delta \quad a : 0, \quad b : 1/2, \quad c : 3, \quad d : 5, \quad e : 7, \quad f : 10$

$\bar{\rho} \quad a : \infty, \quad b : 1, \quad c : 1/6, \quad d : 1/10, \quad e : 1/14, \quad f : 1/20$

# QUANTUM 1D LINEAR THEORY III

The regions of the spectrum that corrisponding to gain $\Im\lambda > 0$ appear like a series of discret line corresponding to different value of n. Each of this line is center in $\bar{\omega} = (2n-1)/2\bar{\rho}$ with distance $1/\bar{\rho}$ and has a width of $4\sqrt{\bar{\rho}}$, this corrispond in the momentum space to shift of $\hbar k/2$ with a width $4\bar{\rho}^{3/2}(\hbar k)$




$\bar{\rho} = 0.1,\ 0.2,\ 0.4$

# FROM 1D TO 3D, PROBLEM AND SOLU

*PROBLEM*

the tranverse motion is essentially classical. So we look for a model which describes the quantum behavior on the longitudinal dimension and at the same time the classical behavior on the tranverse dimension.

- extension via Shroedinger equation $\hat{\psi}(\theta) \to \hat{\psi}(\theta, x, y)$ ?
  NO because it describes a cold beam with a quantum emittance equal to compton vawelength $\frac{\lambda_c}{2\pi}$

*SOLUTION!*

Write a Discrete Wigner function $W_n(\theta, x_t, p_t)$ and perform the classical limit only for the transverse motion in the limit:

$$\alpha \simeq \frac{\lambda_c}{\epsilon_n} \to 0$$

# 1D DISCRETE WIGNER FUNCTION

- In longitudinal dimension $\theta$ is periodic then the momentum of the electron must be discrete
- Therefore we write the 1D Discrete Wigner function introduced for the first time in the (1994) by Bizarro.

$$
\begin{aligned}
W_n(\theta) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta' e^{-i2n\theta} \langle \theta+\theta' \,|\hat{\varrho}|\, \theta-\theta' \rangle \\
&= w_n(\theta) + \sum_{n'}^{\infty} sinc[(n-n'-1/2)\pi] w_{n'+1/2}(\theta) \\
w_s(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta' e^{-i2s\theta} \langle \theta+\theta' \,|\hat{\varrho}|\, \theta-\theta' \rangle, \qquad s = k \; or \; s = k+1/2
\end{aligned}
$$

J.P. Bizarro, PRA 49 3255 (1994)

# 1D WIGNER MODEL FOR QFEL

## Evolution equation

$$\left(\partial_{\bar{z}} + \frac{s}{\bar{\rho}}\partial_\theta\right) w_s - \bar{\rho}\left(Ae^{i\theta} + c.c.\right)\left\{w_{s+1/2} - w_{s-1/2}\right\} = 0$$

$$\partial_{\bar{z}} A + \partial_{z_1} A = \sum_n^\infty \int_{-\pi}^{\pi} d\theta'\, w_{n+1/2}(\theta', \bar{z}, z_1) e^{-i\theta'} + i\delta A$$

For $\hat{\varrho} = |\psi\rangle\langle\psi|$ pure state,

$$\begin{cases} \sum_{n\in Z} W_n(\theta, z) = |\psi(\theta, z)|^2 & \text{Beam profile} \\ \int_{-\pi}^{\pi} W_n(\theta, z) = |c_n(z)|^2 & \text{Density probability} \\ \sum_{n\in Z} \int_{-\pi}^{\pi} W_n(\theta, z) = 1 & \text{Normalization} \end{cases}$$

# CLASSICAL LIMIT

for $\bar{\rho} \gg 1 \rightarrow \bar{p} = s/\bar{\rho}$ becomes a continuos variable

$$\bar{\rho}\left(w_{s+1/2} - w_{s-1/2}\right) \rightarrow \bar{\rho}\left[W\left(\bar{p} + \frac{1}{2\bar{\rho}}\right) - W\left(\bar{p} - \frac{1}{2\bar{\rho}}\right)\right] \rightarrow \partial_{\bar{p}} W$$

In the momentum space $\bar{p} \pm \frac{1}{2\bar{\rho}}$ become $p \pm \frac{\hbar k}{2}$

MAXWELL-WIGNER $\Rightarrow$ MAXWELL-VLASOV

$$\left(\partial_{\bar{z}} + \bar{p}\partial_{\theta}\right) W - \bar{\rho}\left(Ae^{i\theta} + c.c.\right)\partial_{\bar{p}} W = 0$$

$$\left(\partial_{\bar{z}} + \partial_{z_1}\right) A = \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta' \, W(\theta', \bar{p}, \bar{z}, z_1) e^{-i\theta'} + i\delta A$$

# 3D QFEL WIGNER MODEL HOW TO

- Write the 3d quantum Hamiltonian
- Define the statistic operator $\hat{\varrho}$
- Define the 3D Discrete Wigner rappresentation of $\hat{\varrho}$
- Write the 3D Wigner Evolution equations
- Perform the classical limit only on the transverse coordinate
- Or similary:

$$w_s(\theta, \bar{z}, z_1) \rightarrow w_s(\theta, \bar{z}, z_1, \bar{\mathbf{x}}_t, \bar{\mathbf{p}}_t)$$
$$A(\bar{z}, z_1) \rightarrow A(\bar{z}, z_1, \bar{\mathbf{x}}_t), \quad a_w \rightarrow a_w g_l(\bar{\mathbf{x}}_t)$$

$g_l$ = spatial profile of laser

# 3D QFEL HAMILTONIAN

$$\hat{\bar{H}}(\theta, p, \bar{\mathbf{x}}_t, \bar{\mathbf{p}}_t; \bar{z}) = \frac{\hat{p}^2}{2\bar{\rho}} + \frac{\alpha b}{2}\hat{p}_t^2 + p\left[\frac{\xi}{2\rho}(1-|g_l|^2) - \frac{bX}{2}\alpha^2 p_t^2\right] + \frac{\bar{\rho}\xi}{2\rho^2}|g_l|^2 + i\bar{\rho}\left(g_l^* A e^{i\theta} - c.c.\right),$$

$$\bar{\mathbf{x}}_t = \mathbf{x}_t/\sigma, \quad p = \frac{mc(\gamma-\gamma_0)}{\hbar k}, \quad \bar{\mathbf{p}}_t = \frac{mc\gamma_0}{\hbar}\frac{d\bar{\mathbf{x}}_t}{dz}, \quad [\theta, p] = [\bar{x}, \bar{p}_x] =$$

$$b = \frac{L_g}{\beta^*}, \quad \beta^* = \frac{\sigma^2}{\epsilon_r}, \quad X = k\epsilon_r, \quad \xi = \frac{a_w^2}{1+a_w^2}, \quad \alpha = \frac{\hbar}{mc\gamma_0\epsilon_r}.$$

# 3D DISCRETE WIGNER FUNCTION

$$W_m(\theta, \bar{\mathbf{x}}, \bar{\mathbf{p}}_t) = w_m(\theta, \bar{\mathbf{x}}, \bar{\mathbf{p}}_t) + \sum_{n \in Z} \text{sinc}[(m - n - \mu/2)\pi] w_{n+\mu/2}(\theta, \bar{\mathbf{x}}_t, \bar{\mathbf{p}}$$

$$w_s(\theta, \bar{\mathbf{x}}, \bar{\mathbf{p}}_t) = \frac{1}{2\pi^3} \int_{-\pi}^{+\pi} d\theta' e^{-i2\theta' s}$$

$$\cdot \int_{R^2} d\bar{\mathbf{x}}'_t e^{-i2\bar{\mathbf{x}}'_t \bar{\mathbf{p}}_t} \langle \theta - \theta' , \bar{\mathbf{x}}_t + \bar{\mathbf{x}}'_t \,|\, \hat{\varrho} \,|\, \bar{\mathbf{x}}_t - \bar{\mathbf{x}}'_t , \theta + \theta'$$

From the evolution equation $\partial_{\bar{z}} \hat{\varrho} = -i \left[ \hat{H}, \hat{\varrho} \right]$ we obtain

$$a = L_g / Z_r = b / 2X, \quad Z_r = 4\pi\sigma / \lambda$$

$\Downarrow$

# QFEL MAXWELL-WIGNER EQUATION

$$
\begin{aligned}
\partial_{\bar{z}} w_s \quad &+ \quad \left\{ \frac{s}{\bar{\rho}} + \delta + \frac{\xi}{2\rho}(1 - |g_l|^2) - \frac{bX}{2}\bar{\mathbf{p}}_t^2 \right\} \partial_\theta w_s \\
&- \quad \bar{\rho} \left( g_l^* A e^{i\theta} + c.c. \right) \left[ w_{s+1/2} - w_{s-1/2} \right] \\
&+ \quad i\alpha\bar{\rho} \left[ \nabla_{\bar{x}_t} \left( g_l^* A e^{i\theta} - c.c. \right) \right] \left\{ \frac{w_{s+1/2} + w_{s-1/2}}{2} \right\} \quad \boxed{\text{self focusing}} \\
&+ \quad (1 + \alpha X) \left[ b\bar{\mathbf{p}}_t \cdot \nabla_{\bar{\mathbf{x}}_t} - \frac{\xi}{2\rho X} \left( \nabla_{\bar{\mathbf{x}}_t} |g_l|^2 \right) \cdot \nabla_{\bar{\mathbf{p}}_t} \right] w_s = 0,
\end{aligned}
$$

$$
(\partial_{\bar{z}} + \partial_{z_1}) A - ia\nabla^2_{\bar{\mathbf{x}}_t} A = g_l \sum_{m \in Z} \int_{R^2} d^2\bar{\mathbf{p}}_t \int_0^{2\pi} d\theta \, w_{m+1/2}(\theta, \bar{\mathbf{x}}_t, \bar{\mathbf{p}}_t) e^{-i\theta}.
$$

# ANALYSIS OF TRANSVERSE TERM I

$$\dot{\theta} = \left\{ \frac{s}{\bar{\rho}} + \delta + \frac{\xi}{2\rho}(1 - |g_l|^2) - \frac{bX}{2}\bar{\mathbf{p}}_t^2 \right\}$$

Affect the QFEL resonance $\lambda = \frac{\lambda_l}{4\gamma_0^2}\left[1 + a_w^2 + \gamma_0^2(\theta_x^2 + \theta_y^2)\right]$

$$\delta_0 = \frac{\gamma_0 - \gamma_r}{\rho\gamma_0} \rightarrow \left(\frac{\Delta\gamma}{\rho\gamma_0}\right)_{1D} \simeq 1$$

Longitudinal detuning

$$\frac{\xi}{2\rho}(1 - |g_l|^2) \rightarrow \left(\frac{\Delta\gamma}{\rho\gamma_0}\right)_{a_w} \simeq \frac{1}{2\rho}\frac{\Delta a_w^2}{1+a_w^2}$$

Off-resonance due to laser profile variation

$$-\frac{bX}{2}\bar{\mathbf{p}}_t^2 \rightarrow \left(\frac{\Delta\gamma}{\rho\gamma_0}\right)_{x,y} \simeq \frac{1}{2\rho}\frac{\gamma_0^2\bar{\theta}_t^2}{1+a_w^2}$$

Off-resonance due to beam divergence

# ANALYSIS OF TRANSVERSE TERM II

$$\boxed{\dot{\bar{\mathbf{x}}}_t = b\bar{\mathbf{p}}_t}$$

Tranverse motion of the electron (divergenze of beam)

$$\boxed{\dot{\bar{\mathbf{p}}}_t = -\frac{\xi}{2\rho X}\left(\nabla_{\bar{\mathbf{x}}_t}|g_l|^2\right) \rightarrow \dot{\beta} = -\frac{1}{2\gamma_0}\partial_x a_l^2(x)}$$

Ponderomotive force due to the transverse laser profile (defocusing)

# QFEL 3D WORKING EQUATIONS I

$$\boxed{w_s^k = \frac{1}{2\pi}\sum_{k\in Z} w_s^k e^{ik\theta}}$$

$$\begin{aligned}
\partial_{\bar{z}} w_s^k \quad &+ \quad ik\left\{\frac{s}{\bar{\rho}} + \delta + \frac{\xi}{2\rho}(1-|g_l|^2) - \frac{bX}{2}\bar{\mathbf{p}}_t^2\right\}\partial_\theta w_s^k \\
&- \quad \bar{\rho}\left(g_l^* A w_{s+1/2}^{k-1} - g_l^* A w_{s-1/2}^{k-1} + g_l A^* w_{s+1/2}^{k+1} - g_l A^* w_{s-1/2}^{k+1}\right) \\
&+ \quad \left[b\bar{\mathbf{p}}_t \cdot \nabla_{\bar{\mathbf{x}}_t} - \frac{\xi}{2\rho X}\left(\nabla_{\bar{\mathbf{x}}_t}|g_l|^2\right)\cdot\nabla_{\bar{\mathbf{p}}_t}\right] w_s^k = 0,
\end{aligned}$$

$$\left(\partial_{\bar{z}} + \partial_{z_1}\right) A - ia\nabla_{\bar{\mathbf{x}}_t}^2 A = g_l \sum_{m\in Z}\int_{R^2} d^2\bar{\mathbf{p}}_t w_{m+1/2}^1(\theta, \bar{\mathbf{x}}_t, \bar{\mathbf{p}}_t).$$

# QFEL 3D WORKING EQUATIONS II

$$w_n^{2k} \propto \sum_{k\in Z} \langle n+k'|\hat{\varrho}|n-k'\rangle$$

$$w_{n+1/2}^{2k+1} \propto \sum_{k\in Z} \langle n+k'+1|\hat{\varrho}|n-k'\rangle$$

$$\begin{cases} w_n^0 \propto \langle n|\hat{\varrho}|k'\rangle & \text{1d } \rightarrow |c_n|^2 \\ w_{n+1/2}^1 \propto \langle n+1|\hat{\varrho}|n\rangle & \text{1d } \rightarrow c_{n+1}^* c_n = c_n^* c_{n-1} \end{cases}$$

$$\begin{cases} w_0^0 = \langle 0|\hat{\varrho}|0\rangle & \text{1d } \rightarrow |c_0|^2 \\ w_{-1}^0 = \langle -1|\hat{\varrho}|-1\rangle & \text{1d } \rightarrow |c_{-1}|^2 \\ w_{-1/2}^1 = \langle 0|\hat{\varrho}|-1\rangle & \text{1d } \rightarrow c_0^* c_{-1} \end{cases}$$

two level n=0,-1

# 3D QFEL TWO LEVEL APPROXIMATION

$\mathcal{D} = w_0^0 - w_0^{-1}$ Population difference

$\mathcal{B} = w_{-1/2}^1$ Bunching (polarizzation)

$\mathcal{L}_g = L_g/\sqrt{\bar\rho}$ Quntum Gain length

$\bar\rho \lesssim 0.4 \, n = 0, -1$ | $g_l = 1$ static wigler

$$\left(\partial_{z'} + b'\bar{\mathbf{p}}_t \nabla_{\bar{\mathbf{x}}_t}\right)\mathcal{D} + 2\left(\mathcal{A}\mathcal{B}^* + c.c.\right) = 0$$

$$\left(\partial_{z'} + b'\bar{\mathbf{p}}_t \nabla_{\bar{\mathbf{x}}_t} - i\frac{\lfloor X}{2}|\bar{\mathbf{p}}_t|^2\right)\mathcal{B} = \mathcal{A}\mathcal{D}$$

$$\left(\partial_{z'} + \partial_{z'_1} - i\dashv\nabla^2_{\bar{\mathbf{x}}_t}\right)\mathcal{A} = \int_{R^2} d^2\bar{\mathbf{p}}_t\mathcal{B}$$

$\mathcal{A} = A\sqrt{\bar\rho}$ $\quad z' = \bar z\sqrt{\bar\rho} = z/\mathcal{L}_g$ $\quad b' = \mathcal{L}_g/\beta^*$ $\quad a' = \mathcal{L}_g/Z_r$ $\quad z'_1 = z_1\sqrt{\bar\rho}$

# CONCLUSION

- We propose a new 3D Quantum FEL Model based on a MAXWELL-WIGNER equations
- The Wigner approach let us mix the different behaviour between the longitudinal and tranverse dynamics
- The MAXWELL-WIGNER equations switch into the classical MAXWELL-VLASOV equations for $\bar{\rho} \gg 1$
- Future Developments:
  - Implementation 3D Maxwell-Wigner code and simulations
  - Search of particular analytic solution