

A Wigner function model for free electron lasers

N. Piovella^{a,b,*}, M.M. Cola^{a,b}, L. Volpe^{a,b}, R. Gaiba^{b,c}, A. Schiavi^d, R. Bonifacio^{a,e}

^a INFN – Sezione di Milano – via Celoria 16, I-20133 Milano, Italy

^b Dipartimento di Fisica, Università Degli Studi di Milano, via Celoria 16, I-20133 Milano, Italy

^c II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany

^d Dipartimento di Energetica, Università di Roma “La Sapienza” and INFN, Via Scarpa 14, I-00161 Roma, Italy

^e Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brazil

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Abstract

We derive a 1D quantum model for the free electron laser in terms of a Wigner function for the electron beam. We consider both the case of an unbounded space coordinate, for which the momentum is continuous, and the case of a periodic space coordinate, for which the momentum is discrete. The Wigner model extends the Schrödinger model, previously considered, since it also describes the evolution of a mixed state. Furthermore, the Wigner model shows explicitly the classical limit when the quantum FEL parameter $\bar{\rho}$ is large. This model is also the starting point for a future extension to a 3D description of the electron dynamics. The results obtained here are also valid for the quantum description of the collective atomic recoil laser.

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1. Introduction

Free electron laser (FEL) [1] and collective atomic recoil laser (CARL) [2,3] are two examples of collective recoil lasing systems, in which the particles scatter coherently the photons of the pump (the wiggler field in FEL or the laser in CARL) into a forward radiation mode. Exponential enhancement of the emitted radiation and particle self-bunching on the scale of the radiation wavelength are the two main signatures of the collective recoil lasing process. Originally conceived in a semiclassical regime, in which the particle motion is described by classical equations, the collective recoil lasing process allows also for a quantum regime [4–6], in which the particles change their momentum by discrete units of the photon momentum $\hbar k$. Recently, the quantum regime of CARL has been observed with a Bose–Einstein Condensate (BEC) in the

superradiant regime [7–9], in which the condensate scatters the photons of an off-resonant laser recoiling with a momentum multiple of $\hbar k$. A similar regime has been foreseen for a high-gain FEL in the self amplified spontaneous emission (SASE) mode operation, aimed to generate coherent X-ray photons [10,11].

Up to now, the quantum model for FELs and CARLs has been based on a Schrödinger equation for the matter-wave function Ψ , describing the particles, and on the Maxwell equation for the radiation field, coupled in a self-consistent way [12]. The system of equations depends on a single dimensionless parameter $\bar{\rho}$, which represents the maximum number of photons scattered per particle and/or the maximum momentum recoil in units of the photon recoil momentum $\hbar k$. Also, it has been shown that the classical regime is recovered in the limit of large $\bar{\rho}$ [4,5,10]. The Schrödinger–Maxwell equations yields the simplest basic model of the quantum collective recoil lasing mechanism.

In a recent publication, in which we presented a unified quantum description for both FEL and CARL [5], we stated that the Schrödinger equation can be transformed into

* Corresponding author. Address: INFN – Sezione di Milano – via Celoria 16, I-20133 Milano, Italy.

E-mail address: nicola.piovella@mi.infn.it (N. Piovella).

an equation for the Wigner quasi-probability distribution function and we derived some important conclusions from this equation. There are several reasons for describing the particles with a Wigner function W instead of a matter-wave function Ψ : (1) The equation for W shows explicitly the classical limit for $\bar{\rho} \gg 1$; (2) The Wigner function may describe also mixed states, whereas a wave-function Ψ always assumes a pure state, i.e. a perfectly coherent particle sample. Whereas this assumption seems more appropriate for a BEC as in CARL, it does not correspond to the real situation in FELs or in CARL when cold atoms in a thermal state are used. (3) The Wigner function may be extended to a 3D geometry, in which the particles have transverse position and velocity. Especially in FELs, a realistic electron beam has a transverse dimension and an angular divergence much larger than the quantum limit implied by the Heisenberg Uncertainty Principle. For these reasons, it is important to obtain a quantum description of the collective recoil lasing in terms of a Wigner function for the particles.

In this paper we introduce the Wigner function for the particles and we derive its equation for the collective recoil lasing process. We distinguish between the continuous momentum case, in which the space coordinate is unbounded, and the periodic momentum case, in which the space coordinate is periodic. This is the usual assumption in the classical theory of FEL. Whereas for the unbounded case it is possible to use the standard definition of the Wigner function, in the periodic case the space and momentum variables behave as the rotation angle and angular momentum of a rotator, and the periodicity brings to well known difficulties in the quantum description [13].

This paper is organized as follow. In Section 2 we review the classical FEL model and in Section 3 we discuss the quantum FEL model in the classical and quantum regimes. In Section 4a we introduce the continuous Wigner function for the unbounded case and we derive its evolution equation. In Section 4b we define the discrete Wigner function, following the approach introduced by Bizarro [13], we discuss its properties and we derive its evolution equation. In Section 5 we discuss the quantum regime, in which the system is described by only two momentum states. Finally, conclusions are presented in Section 6.

2. Classical FEL model

We start from the classical FEL equations, as formulated in Ref. [1] and written in its standard dimensionless form:

$$\frac{d\theta_j}{d\bar{z}} = \bar{p}_j \quad (1)$$

$$\frac{d\bar{p}_j}{d\bar{z}} = -(Ae^{i\theta_j} + A^*e^{-i\theta_j}) \quad (2)$$

$$\frac{dA}{d\bar{z}} = \frac{1}{N} \sum_{j=1}^N e^{-i\theta_j} + i\delta A, \quad (3)$$

where $\theta_j = (k + k_w)z - ckt_j(z) - \delta\bar{z}$ and $\bar{p}_j = (\gamma_j - \gamma_0)/\rho\gamma_r$ are phase and dimensionless momentum of the j th electron, with $j = 1, \dots, N$, $A = E/\sqrt{mc^2\gamma_0 n\rho}$ is the scaled complex amplitude of the radiation field with electric field E and frequency $\omega = ck$, $\bar{z} = 2k_w\rho z$ is the scaled wiggler length, k_w is the wiggler wavenumber, γ_j is the electron energy (in rest mass units), $\delta = (\gamma_0 - \gamma_r)/\rho\gamma_r$ is the energy detuning, where γ_0 and $\gamma_r = \sqrt{(k/2k_w)(1 + a_w^2)}$ are the initial and the resonant electron energies, $n = N/V$ is the electron density, V is the mode volume, $\rho = (1/\gamma_r)(a_w\omega_p/4ck_w)^{2/3}$ is the classical FEL parameter, $\omega_p = \sqrt{e^2 n/m\epsilon_0}$ is the plasma frequency and a_w is the wiggler parameter. The source term in the field Eq. (3) is the ‘bunching’ $b = \langle \exp(-i\theta) \rangle = (1/N) \sum_j \exp(-i\theta_j)$. The same Eqs. (1)–(3) have been obtained for CARL, using appropriated dimensionless variables and parameters [2,3].

An equivalent fluid description of FEL can be written for the electron distribution function $f(\theta, \bar{p}, \bar{z})$, obeying a Vlasov equation coupled with the equation for A :

$$\frac{\partial f}{\partial \bar{z}} + \bar{p} \frac{\partial f}{\partial \theta} - (Ae^{i\theta} + A^*e^{-i\theta}) \frac{\partial f}{\partial \bar{p}} = 0 \quad (4)$$

$$\frac{dA}{d\bar{z}} = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\bar{p} f(\theta, \bar{p}, \bar{z}) e^{-i\theta} + i\delta A \quad (5)$$

with the normalization condition:

$$\int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\bar{p} f(\theta, \bar{p}, \bar{z}) = 1. \quad (6)$$

Note that in the standard form, the electron position is described by a phase θ varying in $[0, 2\pi)$, whereas the momentum variable \bar{p} is unbounded.

3. Quantum FEL model

The classical model of Eqs. (1)–(3) can be quantized introducing the operators associated to the dimensionless momentum $p_j = \bar{p}_j = mc(\gamma_j - \gamma_0)/\hbar k$ and to the field $a = \sqrt{\bar{\rho}NA}$, where

$$\bar{\rho} = \rho \frac{mc\gamma_r}{\hbar k} \quad (7)$$

is the quantum FEL parameter. Then, the quantum FEL dynamics are described by the following Hamiltonian operator [14]

$$\hat{H} = \sum_{j=1}^N \left[\frac{\hat{p}_j^2}{2\bar{\rho}} + i\sqrt{\frac{\bar{\rho}}{N}} (\hat{a}^\dagger e^{-i\theta_j} - \text{h.c.}) \right] - \delta \hat{a}^\dagger \hat{a}, \quad (8)$$

where $\hat{\theta}_j$ and \hat{p}_j and \hat{a} obey to the commutation rules $[\hat{\theta}_j, \hat{p}_j] = i\delta_{jj}$ and $[\hat{a}, \hat{a}^\dagger] = 1$. The Heisenberg equations for the particle and field operators have been investigated in the linear regime in Ref. [14].

A different approach has been proposed by Preparata [12] which, using the quantum field theory, has shown that the collective dynamics of the system of $N \gg 1$ electrons in an FEL can be described by a single wave function whose

behavior is governed by a Schrödinger-type equation coupled to a self-consistent radiation field equation:

$$i \frac{\partial \Psi}{\partial \bar{z}} = -\frac{1}{2\bar{\rho}} \frac{\partial^2 \Psi}{\partial \theta^2} - i\bar{\rho}[Ae^{i\theta} - \text{c.c.}]\Psi \quad (9)$$

$$\frac{dA}{d\bar{z}} = \int_0^{2\pi} d\theta |\Psi(\theta, \bar{z})|^2 e^{-i\theta} + i\delta A, \quad (10)$$

where $A = a/\sqrt{\bar{\rho}N}$ is a classical field and Ψ is normalized to unity, i.e.

$$\int_0^{2\pi} d\theta |\Psi(\theta, \bar{z})|^2 = 1. \quad (11)$$

Note that Eq. (9) is the Schrödinger equation associated to the single-particle Hamiltonian (8) (with the correspondence $\hat{p} \rightarrow -i\partial_\theta$) and Eq. (10) corresponds to the classical equation for the field when the classical average of $e^{-i\theta}$ is replaced by the quantum ensemble average. In this sense, $|\Psi(\theta, \bar{z})|^2$ may be interpreted as the electron density. Note that the quantum model depends explicitly on the single parameter $\bar{\rho}$, which rules the transition from the classical to the quantum regime.

Eqs. (9) and (10) are conveniently solved in the momentum representation. Assuming that $\Psi(\theta, \bar{z})$ is a periodic function of θ , it can be written as a Fourier series of momentum eigenfunctions $e^{in\theta}$:

$$\Psi(\theta, \bar{z}) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n(\bar{z}) e^{in\theta}, \quad (12)$$

where $|c_n(\bar{z})|^2$ is the probability to have an electron with momentum $p = n(\hbar k)$ at \bar{z} . So inserting Eq. (12) into Eqs. (9) and (10), we obtain [4,5,10]

$$\frac{dc_n}{d\bar{z}} = -i \frac{n^2}{2\bar{\rho}} c_n - \bar{\rho}(Ac_{n-1} - A^*c_{n+1}) \quad (13)$$

$$\frac{dA}{d\bar{z}} = \sum_{n=-\infty}^{\infty} c_n c_{n-1}^* + i\delta A. \quad (14)$$

An analytical and numerical analysis of the solution of Eqs. (13) and (14), with the particles initially in a single momentum state with $n = 0$ (i.e. $c_n(0) = \delta_{n0}$), shows that the system behaves classically for $\bar{\rho} \gg 1$, i.e. the solution coincides with that of the classical Eqs. (1)–(3) [4,5]. Conversely, in the quantum limit $\bar{\rho} \leq 1$ the particles occupy only the first adjacent momentum level $n = -1$ and behave as a two-level system interacting with the radiation mode.

4. A Wigner function for FEL

4.1. Continuous case

In order to obtain for the FEL a quantum description analog to the classical Vlasov Eq. (4), we must use the Wigner distribution function [5]. We start with the standard definition of the Wigner function for a state with statistical operator $\hat{\rho}(\bar{z}) = \sum_\lambda p_\lambda |\Psi_\lambda\rangle\langle\Psi_\lambda|$, where the space coordinate θ is assumed to be unbounded:

$$W(\theta, p, \bar{z}) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \langle \theta + \theta' | \hat{\rho}(\bar{z}) | \theta - \theta' \rangle. \quad (15)$$

For a pure state $\hat{\rho}(\bar{z}) = |\Psi\rangle\langle\Psi|$ and

$$|\Psi(\theta, \bar{z})|^2 = \int_{-\infty}^{+\infty} dp W(\theta, p, \bar{z}). \quad (16)$$

As it is shown in Appendix A, it is possible to derive from Eq. (9) the following equation for the Wigner function

$$\begin{aligned} \frac{\partial W(\theta, p, \bar{z})}{\partial \bar{z}} + \frac{p}{\bar{\rho}} \frac{\partial W(\theta, p, \bar{z})}{\partial \theta} - \bar{\rho}(Ae^{i\theta} + A^*e^{-i\theta}) \\ \times \left\{ W\left(\theta, p + \frac{1}{2}, \bar{z}\right) - W\left(\theta, p - \frac{1}{2}, \bar{z}\right) \right\} = 0, \end{aligned} \quad (17)$$

coupled with the equation for the radiation field,

$$\frac{dA}{d\bar{z}} = \int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty} dp W(\theta, p, \bar{z}) e^{-i\theta} + i\delta A. \quad (18)$$

Introducing $\bar{p} = p/\bar{\rho}$, Eq. (17) becomes

$$\begin{aligned} \frac{\partial W(\theta, \bar{p}, \bar{z})}{\partial \bar{z}} + \bar{p} \frac{\partial W(\theta, \bar{p}, \bar{z})}{\partial \theta} - \bar{\rho}(Ae^{i\theta} + A^*e^{-i\theta}) \\ \times \left\{ W\left(\theta, \bar{p} + \frac{1}{2\bar{\rho}}, \bar{z}\right) - W\left(\theta, \bar{p} - \frac{1}{2\bar{\rho}}, \bar{z}\right) \right\} = 0. \end{aligned} \quad (19)$$

In the limit $\bar{\rho} \rightarrow \infty$ the finite difference term in Eq. (19) becomes the derivative of W with respect to \bar{p} ,

$$\lim_{\bar{\rho} \rightarrow \infty} \bar{\rho} \left\{ W\left(\theta, \bar{p} + \frac{1}{2\bar{\rho}}, \bar{z}\right) - W\left(\theta, \bar{p} - \frac{1}{2\bar{\rho}}, \bar{z}\right) \right\} = \frac{\partial W}{\partial \bar{p}}, \quad (20)$$

so that the equation for the Wigner function becomes the Vlasov Eq. (4), where however the space coordinate θ is unbounded, whereas in Eqs. (4) and (5) the classical distribution function $f(\theta, \bar{p})$ is periodic in $(0, 2\pi)$. Although the choice of the θ -domain in the classical picture has no consequences on the momentum variable \bar{p} , in the quantum description they are intrinsically related, since if θ is a periodic variable in $(0, 2\pi)$, then necessarily the conjugated momentum variable p is discrete. This makes necessary to introduce a discrete Wigner function.

4.2. Discrete case

For variables such as rotation angle and angular momentum, well known difficulties arise due to periodicity [15]. To solve this problem, it is possible to define a discrete Wigner function, following the work of Bizarro [13]

$$W_m(\theta, \bar{z}) = \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} d\theta' e^{-2im\theta'} \langle \theta + \theta' | \hat{\rho}(\bar{z}) | \theta - \theta' \rangle. \quad (21)$$

The momentum is now represented by the discrete label m . This definition keeps the required properties of the Wigner function as a quasi-probability distribution. For simplicity, we assume in the following a pure state (i.e. $\hat{\rho}(\bar{z}) = |\Psi\rangle\langle\Psi|$). By tracing over one variable, we obtain the probability distribution for the other

$$\int_{-\pi}^{+\pi} d\theta W_m(\theta, \bar{z}) = |c_m(\bar{z})|^2 \quad (22)$$

$$\sum_{m=-\infty}^{+\infty} W_m(\theta, \bar{z}) = |\Psi(\theta, \bar{z})|^2. \quad (23)$$

This implies the normalization of the discrete Wigner function

$$\sum_{m=-\infty}^{+\infty} \int_{-\pi}^{+\pi} d\theta W_m(\theta, \bar{z}) = 1. \quad (24)$$

Inserting in (21) the Fourier expansion (12) we obtain

$$W_m(\theta, \bar{z}) = \frac{1}{2\pi} \sum_{m', m''} c_{m'}^*(t) c_{m''}(t) e^{-i(m' - m'')\theta} \text{sinc}\left[\frac{(2m - m' - m'')\pi}{2}\right]. \quad (25)$$

Following Ref. [13], we write

$$W_m(\theta, \bar{z}) = w_m(\theta, \bar{z}) + \sum_{m'=-\infty}^{+\infty} \frac{(-1)^{m-m'-1}}{(m - m' - \frac{1}{2})\pi} w_{m'+1/2}(\theta, \bar{z}), \quad (26)$$

where

$$w_m(\theta, \bar{z}) = \frac{1}{2\pi} \sum_{m'=-\infty}^{+\infty} c_{m+m'}^*(\bar{z}) c_{m-m'}(\bar{z}) e^{-i2m'\theta} \quad (27)$$

$$w_{m+1/2}(\theta, \bar{z}) = \frac{1}{2\pi} \sum_{m'=-\infty}^{+\infty} c_{m+m'+1}^*(\bar{z}) c_{m-m'}(\bar{z}) e^{-i(2m'+1)\theta}. \quad (28)$$

The introduction of the two new functions $w_n(\theta)$ and $w_{n+1/2}(\theta)$ is necessary in order to obtain a dynamical equation for the Wigner function. The integer and half-integer functions $w_n(\theta)$ and $w_{n+1/2}(\theta)$ are orthogonal to each other

$$\int_{-\pi}^{+\pi} d\theta w_m(\theta, \bar{z}) w_{n+1/2}(\theta, \bar{z}) = 0, \quad (29)$$

for all m, n , and contain all the information needed to determine $W_m(\theta, \bar{z})$. In particular, the probabilities for the momentum m and the phase θ can be derived directly from $w_m(\theta)$ and $w_{m+1/2}(\theta)$

$$|c_m(\bar{z})|^2 = \int_{-\pi}^{+\pi} d\theta w_m(\theta, \bar{z}) \quad (30)$$

$$|\Psi(\theta, \bar{z})|^2 = \sum_{m=-\infty}^{+\infty} \{w_m(\theta, \bar{z}) + w_{m+1/2}(\theta, \bar{z})\}, \quad (31)$$

so that the normalization condition is

$$\sum_{m=-\infty}^{+\infty} \int_{-\pi}^{+\pi} d\theta w_m(\theta, \bar{z}) = 1. \quad (32)$$

The importance of these functions is that, while for the $W_m(\theta, \bar{z})$ it is not possible to find a closed evolution equation, it can be done for $w_m(\theta, \bar{z})$ and $w_{m+1/2}(\theta, \bar{z})$. Deriving Eqs. (27) and (28) with respect to \bar{z} and inserting Eqs. (13) for the amplitudes $c_n(\bar{z})$, we obtain, after some algebra,

$$\frac{\partial w_s(\theta, \bar{z})}{\partial \bar{z}} + \frac{s}{\bar{\rho}} \frac{\partial w_s(\theta, \bar{z})}{\partial \theta} - \bar{\rho}(Ae^{i\theta} + A^*e^{-i\theta})\{w_{s+1/2}(\theta, \bar{z}) - w_{s-1/2}(\theta, \bar{z})\} = 0, \quad (33)$$

where $s = m$ or $s = m + 1/2$. Eq. (33) is similar to the Wigner Eq. (17) obtained for the continuous case, with the difference that now the momentum is discrete and two separate distribution functions are needed. In this new formalism the bunching of the electron beam can be written as

$$\langle e^{-i\theta} \rangle = \sum_{n=-\infty}^{\infty} c_n c_{n-1}^* = \sum_{m=-\infty}^{+\infty} \int_{-\pi}^{+\pi} d\theta e^{-i\theta} w_{m+1/2}(\theta, \bar{z}), \quad (34)$$

so that Eq. (33) can be closed by coupling it to the one for the radiation field

$$\frac{dA}{d\bar{z}} = \sum_{m=-\infty}^{+\infty} \int_{-\pi}^{+\pi} d\theta e^{-i\theta} w_{m+1/2}(\theta, \bar{z}) + i\delta A. \quad (35)$$

Eqs. (33) and (35), in the case of the pure state of Eq. (12), are equivalent to Eqs. (13) and (14). Note that, in the limit $\bar{\rho} \rightarrow \infty$, $\bar{p} = s/\bar{\rho}$ becomes a continuous variable, $w_s(\theta) \rightarrow \bar{\rho} f(\theta, \bar{p})$ and $\bar{\rho}[w_{s+1/2}(\theta) - w_{s-1/2}(\theta)] \rightarrow \partial f / \partial \bar{p}$, in the same way as in Eq. (20) for the continuous case. Hence, for $\bar{\rho} \gg 1$ Eq. (33) reduces to the corresponding Vlasov Eq. (4) for the classical distribution $f(\theta, \bar{p})$, with $\bar{p} \in (-\infty, +\infty)$ and $\theta \in (0, 2\pi]$, and Eq. (35) reduces to Eq. (5).

Since $w_s(\theta, \bar{z})$ is periodic in θ , it can be expanded in a Fourier series:

$$w_s(\theta, \bar{z}) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} w_s^k(\bar{z}) e^{ik\theta}. \quad (36)$$

Using (36), Eqs. (33) and (35) become:

$$\frac{\partial w_s^k}{\partial \bar{z}} + ik \frac{s}{\bar{\rho}} w_s^k - \bar{\rho}[A(w_{s+1/2}^{k-1} - w_{s-1/2}^{k-1}) + A^*(w_{s+1/2}^{k+1} - w_{s-1/2}^{k+1})] = 0 \quad (37)$$

$$\frac{dA}{d\bar{z}} = \sum_{m=-\infty}^{+\infty} w_{m+1/2}^1 + i\delta A. \quad (38)$$

The Fourier components $w_s^k(\bar{z})$ are related to the Fourier components $c_m(\bar{z})$ of the wave function $\Psi(\theta, \bar{z})$, in the following way

$$w_m^{2k} = c_{m+k}^* c_{m-k} \quad (39)$$

$$w_{m+1/2}^{2k+1} = c_{m+k+1}^* c_{m-k}. \quad (40)$$

In particular, $w_m^0 = |c_m|^2$ are the momentum probabilities and $w_{m+1/2}^1 = c_{m+1}^* c_m$ are the m -th bunching components, describing the overlapping between the m and $m+1$ states. A numerical analysis has shown full agreement between the solutions of Eqs. (37), (38) and Eqs. (13), (14). Fig. 1 shows the intensity $\bar{\rho}|A|^2$ and the bunching $|b| = |\sum_m w_{m+1/2}^1|$ (Fig. 1a) and the Fourier coefficients w_0^0 , w_{-1}^0 and $|w_{-1/2}^1|$ (Fig. 1b) vs. $z' = \sqrt{\bar{\rho}z}$ for $\bar{\rho} = 0.1$ and $\delta = 5$ (quantum regime). The initial conditions are $w_0^0(0) = 1 - \epsilon^2$, $w_{-1}^0(0) = \epsilon^2$ and $w_{-1/2}^1(0) = \epsilon\sqrt{1 - \epsilon^2}$, where $\epsilon = 10^{-2}$. In this regime only two momentum states, $m = 0$ and $m = -1$, are significantly populated. The crosses in Fig. 1a represent the intensity $\bar{\rho}|A|^2$ and the bunching

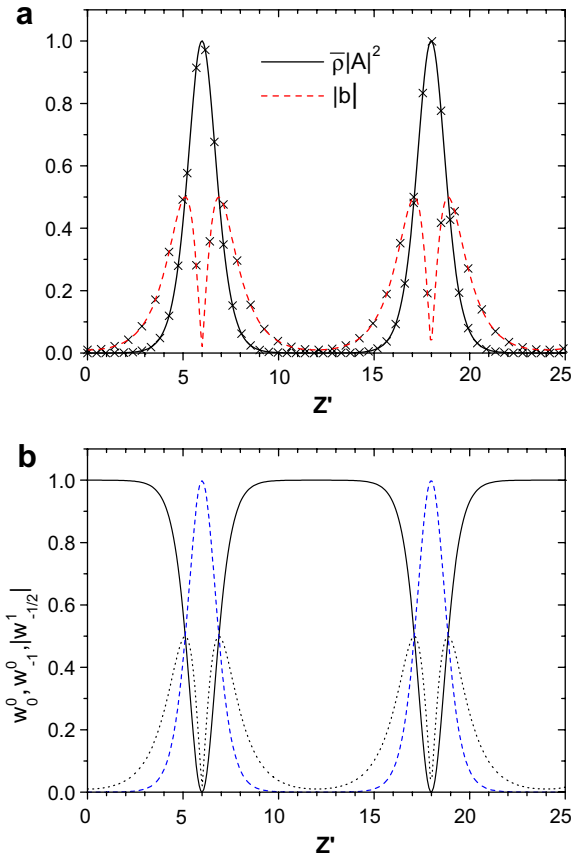


Fig. 1. Quantum regime, for $\bar{\rho} = 0.1$ and $\delta = 5$. (a) Intensity $\bar{\rho}|A|^2$ (continuous line) and bunching $|b|$ (dashed line) vs. $z' = \sqrt{\bar{\rho}}z$. The crosses represent the solution of Eqs. (13) and (14). (b) w_0^0 (continuous line), w_{-1}^0 (dashed line) and $|w_{-1/2}^1|$ (dotted line) vs. z' .

$|b| = |\sum_m c_m c_{m-1}^*|$, as calculated from the solution of Eqs. (13) and (14). Fig. 2 shows $|A|^2$ and b vs. \bar{z} for $\bar{\rho} = 5$ and $\delta = 0$ (classical regime). In both the quantum and classical regime the two solutions overlap perfectly. Finally, Fig. 3 shows a bi-dimensional representation of w_s^k as a function

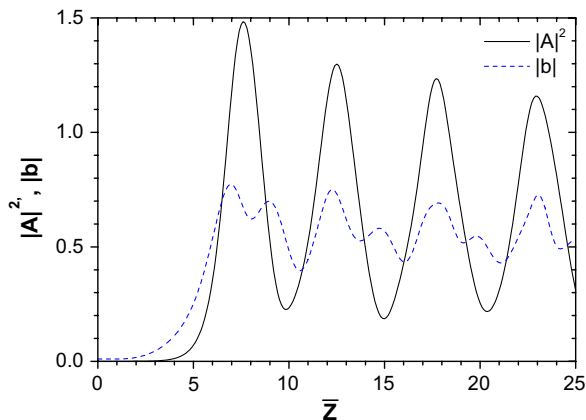


Fig. 2. Classical regime, for $\bar{\rho} = 5$, $\delta = 0$. Intensity $|A|^2$ (continuous line) and bunching $|b|$ (dashed line) vs. \bar{z} .

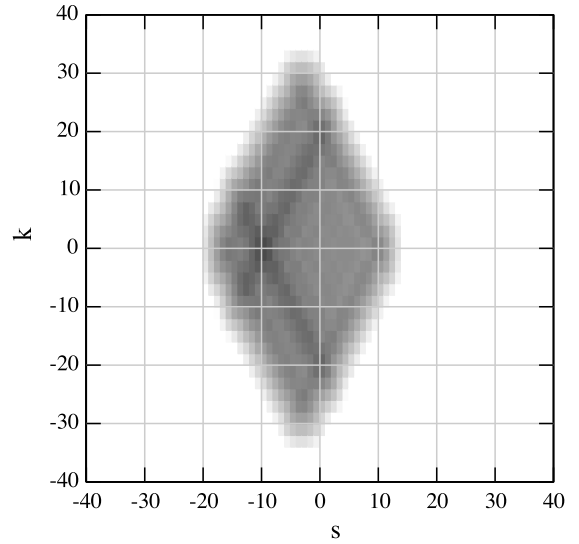


Fig. 3. w_s^k vs. s and k at $\bar{z} = 7.5$, for $\bar{\rho} = 5$ and $\delta = 0$.

of $s = n/2$ (with $n = 0, \pm 1, \dots$) and $k = 0, \pm 1, \dots$ for the classical case of Fig. 2 at $\bar{z} = 7.5$, corresponding to the position of the peak intensity. We observe that in the classical regime a large number of momentum states (approximately +12) and Fourier component (up to $|k| \leq 32$) becomes occupied. On the contrary, in the quantum regime $\bar{\rho} \ll 1$, only w_0^0 , w_{-1}^0 and $w_{-1/2}^1$ (corresponding to the momentum states $m = 0, -1$) are appreciably different from zero.

5. Two-level approximation

It is interesting to see which is the form of the Wigner function in the quantum regime $\bar{\rho} \ll 1$, for which the momentum space is spanned only by the two states $m = 0$ and $m = -1$, i.e. the wave function is $\Psi(\theta, \bar{z}) = (1/\sqrt{2\pi})[c_0(\bar{z}) + c_{-1}(\bar{z})e^{-i\theta}]$. Then, from (26) to (28) we obtain

$$W_m(\theta) = \frac{1}{2\pi} \left\{ w_m^0 + \frac{(-1)^m}{(m+1/2)\pi} [w_{-1/2}^1 e^{i\theta} + c.c.] \right\}, \quad (41)$$

where $w_m = 0$ for $m \neq 0, -1$. We note that we have an infinite number of component W_m , also if we have only two momentum states.

We define the population difference $D = w_0^0 - w_{-1}^0$ and the polarization $B = w_{-1/2}^1$ which, from Eqs. (37) and (38), evolve, together with the radiation field A , with the following equations:

$$\frac{dD}{d\bar{z}} = -2\bar{\rho}(AB^* + A^*B) \quad (42)$$

$$\frac{dB}{d\bar{z}} = \frac{i}{2\bar{\rho}}B + \bar{\rho}DA \quad (43)$$

$$\frac{dA}{d\bar{z}} = B + i\delta A. \quad (44)$$

Note that the $\bar{\rho}$ parameter can be reabsorbed through the redefinition [10]

$$A' = \sqrt{\bar{\rho}} A e^{-i\delta \bar{z}} \quad (45)$$

$$B' = B e^{-i\delta \bar{z}} \quad (46)$$

$$z' = \sqrt{\bar{\rho}} \bar{z}, \quad (47)$$

so that the equations take the form:

$$\frac{dD}{dz'} = -2(A'B'^* + A'^*B') \quad (48)$$

$$\frac{dB'}{dz'} = -i\delta' B' + DA' \quad (49)$$

$$\frac{dA'}{dz'} = B', \quad (50)$$

where $\delta' = [\delta - 1/(2\bar{\rho})]/\sqrt{\bar{\rho}}$. At resonance, $\delta' = 0$, the solution of Eqs. (48) and (49) for an initial condition close to $D(0) = 1$ is $A'(z') = \text{sech}(z' - z'_0)$, $B'(z') = -\sinh(z' - z'_0) \text{sech}^2(z' - z'_0)$ and $D(z') = 1 - 2\text{sech}^2(z' - z'_0)$, where z'_0 depends on the initial fluctuation of polarization B' [16]. So, the Wigner function (41) for $m = 0, -1$ becomes

$$W_0(\theta') = \frac{1}{2\pi} \left\{ 1 - \text{sech}^2(z' - z'_0) \left[1 + \frac{4}{\pi} \sinh(z' - z'_0) \cos(\theta') \right] \right\} \quad (51)$$

$$W_{-1}(\theta') = \frac{1}{2\pi} \text{sech}^2(z' - z'_0) \left\{ 1 - \frac{4}{\pi} \sinh(z' - z'_0) \cos(\theta') \right\}, \quad (52)$$

where $\theta' = \theta - \bar{z}/(2\bar{\rho})$. For a classical system, the Wigner function is positive in all the points of the phase space and assumes the role of the probability density distribution, while for a quantum system this does not hold and the Wigner function can become negative in certain zones of the phase space. It's then interesting to find out when $W_{0,-1}$ becomes negative, i.e. when the system can be certainly considered in a quantum regime.

Using the integral of motion $D^2 + 4B'^2 = 1$, it's easy to see that

$$W_0(\theta', z') < 0 \quad \text{when} \quad \frac{w_{-1}^0}{w_0^0} > \alpha_\theta^2 \quad (53)$$

$$W_{-1}(\theta', z') < 0 \quad \text{when} \quad \frac{w_{-1}^0}{w_0^0} < \frac{1}{\alpha_\theta^2}, \quad (54)$$

where $\alpha_\theta = \pi/(4 \cos \theta')$. Since $w_0^0 + w_{-1}^0 = 1$, Eq. (53) and (54) can be written in the compact form

$$W_m(\theta', z') < 0 \quad \text{when} \quad w_m^0 < \frac{1}{1 + \alpha_\theta^2}, \quad (55)$$

where $m = 0, -1$. From Eq. (55), it follows that the Wigner functions $W_{0,-1}(\theta', \bar{z})$ can be negative only when the population difference is $|D| < (16 - \pi^2)/(16 + \pi^2) \approx 0.237$ or equivalently when $B' > 4\pi/(16 + \pi^2) \approx 0.486$, i.e. when the polarization between the two states is close to its maximum value 0.5.

6. Conclusions

We have derived a 1D quantum FEL model in terms of a Wigner function for the electron beam. This model extends the Schrödinger–Maxwell model discussed in

previous works [4,6,10,11], where the electron beam is described by a collective wave function $\Psi(\theta, \bar{z})$. In fact, the Wigner function model has a broader validity than the Schrödinger equation, since it can also describe a statistical mixture of states, which can not be represented by a wave function but rather by a density operator [17]. In a first step, we have derived the equation for a continuous Wigner function, published in Ref. [5] without demonstration. However, the continuous Wigner function cannot be defined for a periodic system, as the FEL is usually considered, in which the electron coordinate is the phase of the periodic ponderomotive potential. In order to define properly the Wigner function for a periodic spatial variable, we have then introduced a discrete Wigner function and we have derived a set of equations which is equivalent, in the case of the pure state considered in Section 4, to the Schrödinger–Maxwell model. The Wigner function model has the important property that it shows explicitly the classical limit for $\bar{\rho} \gg 1$, in which a finite difference term becomes a continuous derivative and the equation for the Wigner function becomes a classical Vlasov equation. This property shows the transition from the quantum to the classical regime. Furthermore, the 1D Wigner model is the starting point for the extension to a 3D description of the electron dynamics, as it will be discussed in a future publication. This can be of relevance for a realistic description of the quantum FEL regime, in which the transverse motion of the electrons is expected to reduce the gain and the emission process.

Appendix A. Derivation of Eq. (17)

The Wigner function for the continuous case and a pure state is defined as:

$$W(\theta, p, \bar{z}) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \Psi^*(\theta - \theta', \bar{z}) \Psi(\theta + \theta', \bar{z}). \quad (A.1)$$

Setting $\theta_\pm = \theta \pm \theta'$ and differentiating with respect to \bar{z} , we obtain

$$\frac{\partial W(\theta, p, \bar{z})}{\partial \bar{z}} = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\theta' \exp(-i2p\theta') \left\{ \frac{\partial \Psi^*(\theta_-, \bar{z})}{\partial \bar{z}} \Psi(\theta_+, \bar{z}) + \Psi^*(\theta_-, \bar{z}) \frac{\partial \Psi(\theta_+, \bar{z})}{\partial \bar{z}} \right\}. \quad (A.2)$$

Using Eq. (9) we can write

$$\begin{aligned} \frac{\partial \Psi(\theta_+, \bar{z})}{\partial \bar{z}} &= \frac{i}{2\bar{\rho}} \frac{\partial^2 \Psi(\theta_+, \bar{z})}{\partial \theta_+^2} - \bar{\rho} (Ae^{i\theta_+} - c.c.) \Psi(\theta_+, \bar{z}) \\ \frac{\partial \Psi^*(\theta_-, \bar{z})}{\partial \bar{z}} &= -\frac{i}{2\bar{\rho}} \frac{\partial^2 \Psi^*(\theta_-, \bar{z})}{\partial \theta_-^2} + \bar{\rho} (Ae^{i\theta_-} - c.c.) \Psi^*(\theta_-, \bar{z}), \end{aligned}$$

which once inserted in (A.2) yield

$$\frac{\partial W(\theta, p, \bar{z})}{\partial \bar{z}} = \mathcal{A} + \mathcal{B} \quad (A.3)$$

where

$$\mathcal{A} = \frac{i}{2\pi\bar{\rho}} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \times \left\{ \frac{\partial^2 \Psi(\theta_+, \bar{z})}{\partial \theta_+^2} \Psi^*(\theta_-, \bar{z}) - \frac{\partial^2 \Psi^*(\theta_-, \bar{z})}{\partial \theta_-^2} \Psi(\theta_+, \bar{z}) \right\} \quad (\text{A.4})$$

$$\mathcal{B} = \frac{\bar{\rho}}{\pi} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \times \{A(e^{i\theta_-} - e^{i\theta_+}) - c.c.\} \Psi^*(\theta_-, \bar{z}) \Psi(\theta_+, \bar{z}). \quad (\text{A.5})$$

Let's first calculate \mathcal{A} . Since $\partial\theta'/\partial\theta_{\pm} = \pm 1$, with a partial integration with respect to θ' , we obtain

$$\begin{aligned} \mathcal{A} &= \frac{i}{2\pi\bar{\rho}} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \left\{ \frac{\partial^2 \Psi(\theta_+, \bar{z})}{\partial \theta'^2} \Psi^*(\theta_-, \bar{z}) - \frac{\partial^2 \Psi^*(\theta_-, \bar{z})}{\partial \theta'^2} \Psi(\theta_+, \bar{z}) \right\} \\ &= \frac{p}{\pi\bar{\rho}} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \left\{ \frac{\partial \Psi^*(\theta_-, \bar{z})}{\partial \theta'} \Psi(\theta_+, \bar{z}) - \frac{\partial \Psi(\theta_+, \bar{z})}{\partial \theta'} \Psi^*(\theta_-, \bar{z}) \right\} \\ &= -\frac{p}{\pi\bar{\rho}} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \left\{ \frac{\partial \Psi^*(\theta_-, \bar{z})}{\partial \theta_-} \Psi(\theta_+, \bar{z}) + \frac{\partial \Psi(\theta_+, \bar{z})}{\partial \theta_+} \Psi^*(\theta_-, \bar{z}) \right\}. \end{aligned} \quad (\text{A.6})$$

Since $\partial\theta_{\pm}/\partial\theta = 1$,

$$\begin{aligned} \mathcal{A} &= -\frac{p}{\pi\bar{\rho}} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \left\{ \frac{\partial \Psi^*(\theta_-, \bar{z})}{\partial \theta} \Psi(\theta_+, \bar{z}) + \frac{\partial \Psi(\theta_+, \bar{z})}{\partial \theta} \Psi^*(\theta_-, \bar{z}) \right\} \\ &= -\frac{p}{\pi\bar{\rho}} \frac{\partial}{\partial \theta} \int_{-\infty}^{+\infty} d\theta' e^{-i2p\theta'} \{ \Psi^*(\theta_-, \bar{z}) \Psi(\theta_+, \bar{z}) \} \\ &= -\frac{p}{\bar{\rho}} \frac{\partial W(\theta, p, \bar{z})}{\partial \theta}. \end{aligned} \quad (\text{A.7})$$

Let's now calculate \mathcal{B} . Rearranging the terms in Eq. (A.5), we can write:

$$\mathcal{B} = \frac{\bar{\rho}}{\pi} (Ae^{i\theta} + c.c.) \int_{-\infty}^{+\infty} d\theta' [e^{-i2(p+1/2)\theta'} - e^{-i2(p-1/2)\theta'}] \Psi^*(\theta_-, \bar{z}) \Psi(\theta_+, \bar{z}) \quad (\text{A.8})$$

and using the definition (A.1), we have

$$\mathcal{B} = \bar{\rho} (Ae^{i\theta} + c.c.) \left\{ W\left(\theta, p + \frac{1}{2}, \bar{z}\right) - W\left(\theta, p - \frac{1}{2}, \bar{z}\right) \right\}. \quad (\text{A.9})$$

Finally, we obtain Eq. (17):

$$\begin{aligned} \frac{\partial W(\theta, p, \bar{z})}{\partial \bar{z}} &= -\frac{p}{\bar{\rho}} \frac{\partial W(\theta, p, \bar{z})}{\partial \theta} + \bar{\rho} (Ae^{i\theta} + c.c.) \\ &\times \left\{ W\left(\theta, p + \frac{1}{2}, \bar{z}\right) - W\left(\theta, p - \frac{1}{2}, \bar{z}\right) \right\}. \end{aligned} \quad (\text{A.10})$$

The same Eq. (A.10) for the more general Wigner function (15) can be obtained in the case of a mixed state, due to the linearity of the statistical operator, $\hat{\rho}(\bar{z}) = \sum_{\lambda} p_{\lambda} |\Psi_{\lambda}\rangle \langle \Psi_{\lambda}|$.

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