3D WIGNER FUNCTION MODEL FOR A QUANTUM FREE ELECTRON LASER

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Outline

1. From the classical to the quantum FEL description
2. Schrödinger vs. Wigner models in 1D
3. From 1D to 3D quantum FEL model with a laser wiggler
4. 3D Wigner model
5. Discussion and results

CLASSICAL FEL 1D MODEL

$$\frac{\partial^2 \theta_j}{\partial \bar{z}^2} = -(Ae^{i\theta_j} + c.c.)$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j}$$

R.Bonifacio, C.Pellegrini, L.Narducci, Opt. Comm. (1984) (steady-state)

$$\bar{z} = z / L_g$$

$$L_g = \lambda_w / 4\pi\rho$$

gain length

$$z_1 = (z - v_r t) / L_c$$

$$L_c = \lambda_r / 4\pi\rho$$

cooperation length

$$\theta_j = (k + k_w)z - \omega t_j;$$

$$\rho |A|^2 = \frac{P_{rad}}{P_{beam}}$$

$$\rho \propto \frac{1}{\gamma_R}\left(\frac{a_w \lambda_w}{\sigma}\right)^{2/3} I^{1/3}$$

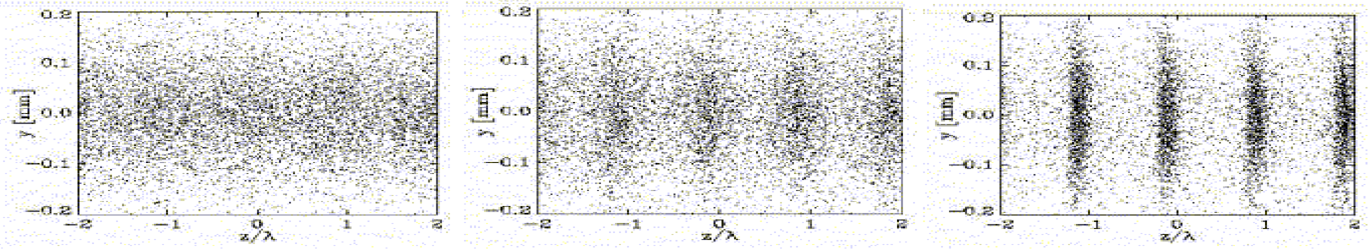
FEL parameter

HIGH-GAIN REGIME

- exponential growth of intensity and bunching
- start up from noise
- saturation at *A*~1 ($P_{rad} \sim \rho\, P_{beam}$) after several L_g

b~0 → b~0.8

bunching:



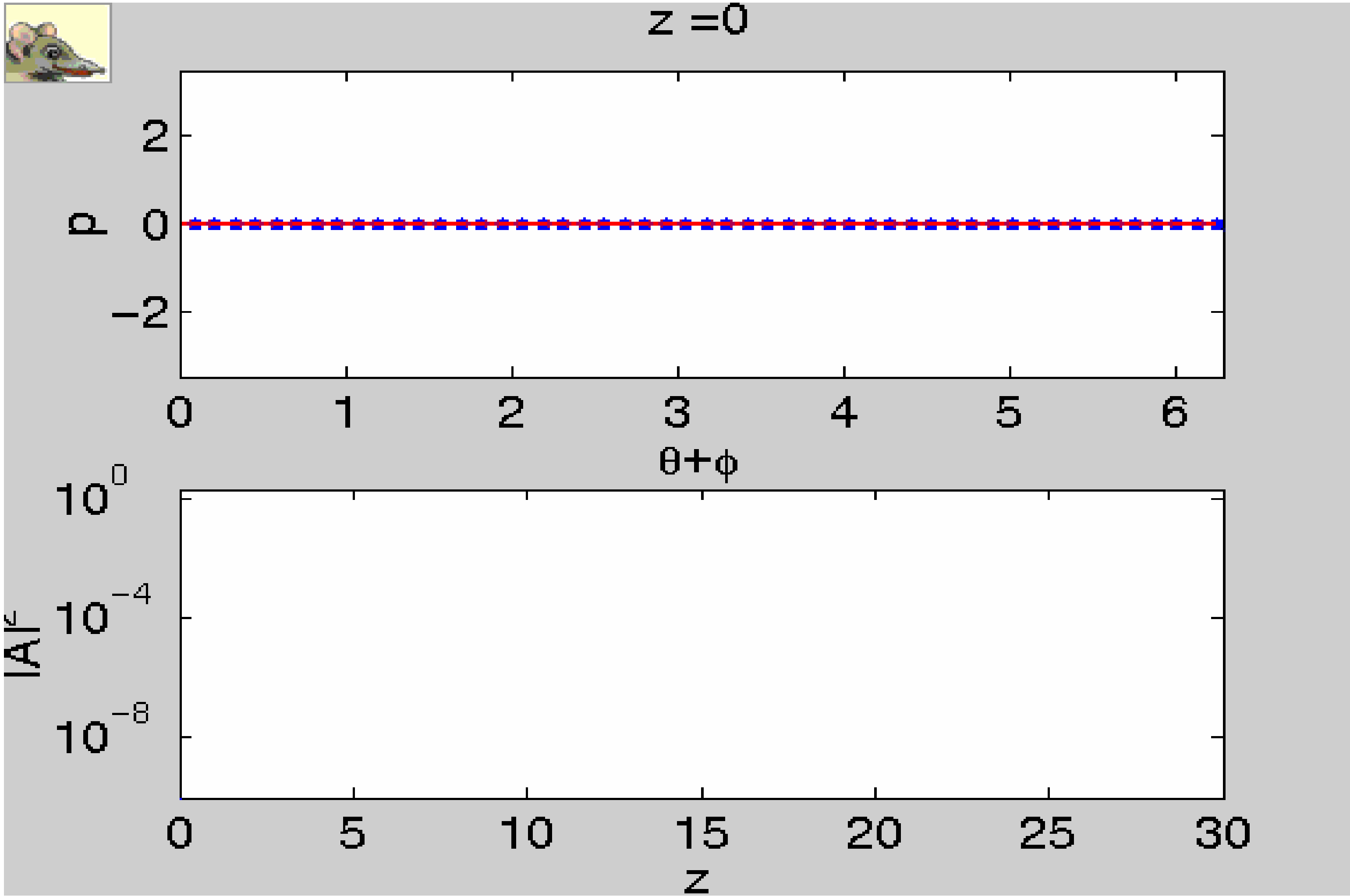
$$b = \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j}$$

wiggler length (several L_g)

neglecting SLIPPAGE: $\cancel{\frac{\partial A}{\partial z_1}}$ (STEADY-STATE regime)

electrons behave as coupled pendula in a self-consistent potential:

$$\boxed{V(\theta+\varphi)=2|A|\cos(\theta+\varphi)}$$



$$p=\frac{\gamma-\gamma_r}{\rho\gamma_r}$$

Simulation
by G.R.M Robb

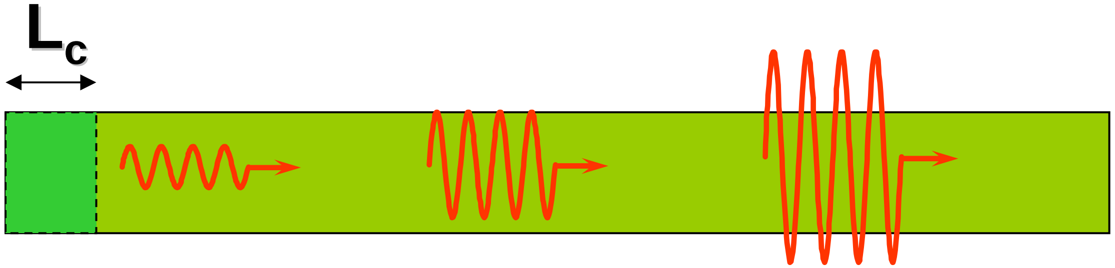
SLIPPAGE EFFECT: SUPERRADIANCE

R. Bonifacio, B.W. McNeil, P. Pierini PRA (1989)

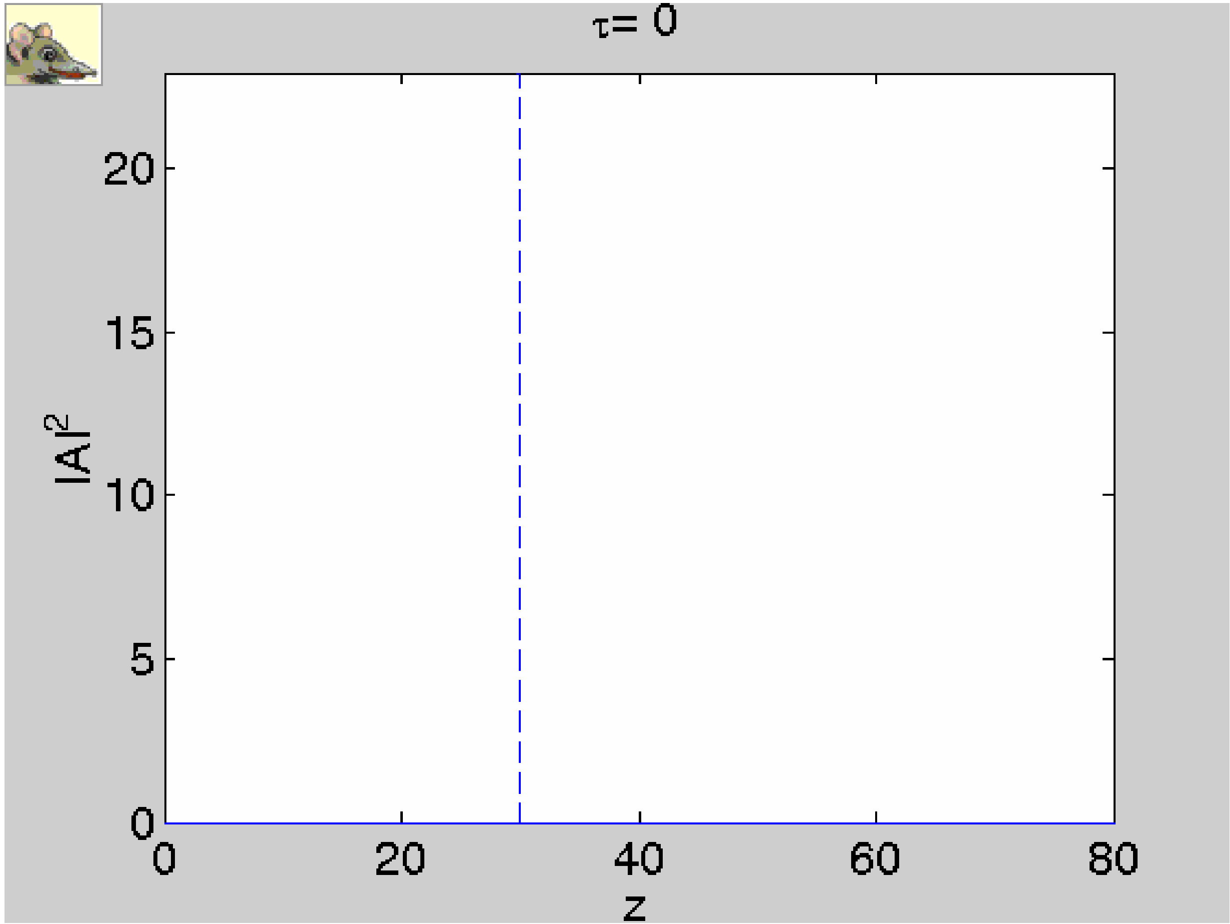
Radiation propagates faster than electrons and advance them by a slippage length $(\lambda_r/\lambda_w)L_w = \lambda_r N_w$

Electrons in a cooperation length $L_c=\lambda_r/4\pi\rho$ radiate a SUPERRADIANT (SR) PULSE.

if $L_b >> L_c$ the SR pulse gets amplified as it propagates forward through the beam with no saturation.



STEADY-STATE + SUPERRADIANT SPIKE ('second instability') from a resonant coherent seed



L_b=30 L_c

0<z<50 L_g

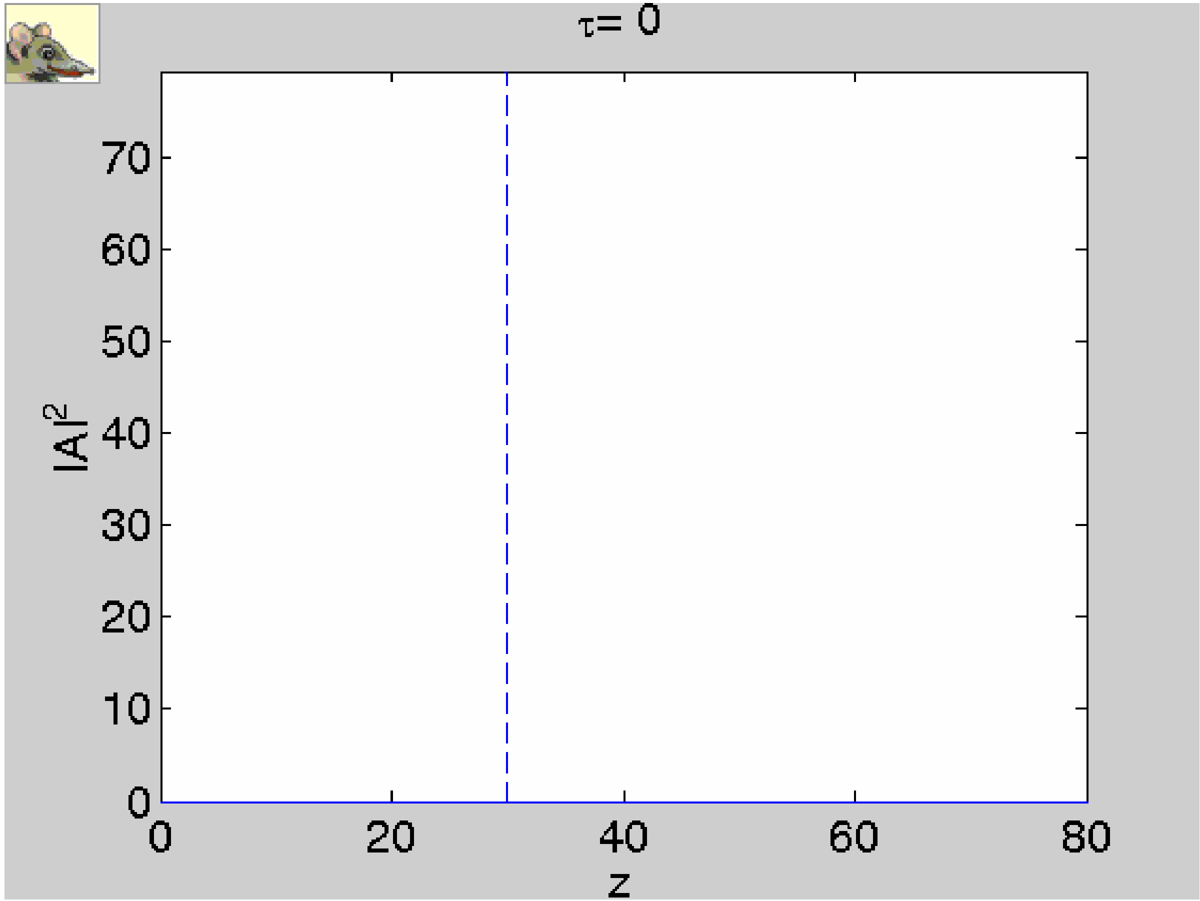
δ=0

$$\left(z_1 = \frac{z - vt}{L_c} \right)$$

simulation
by G.R.M Robb

SUPERRADIANT SPIKE
in a detuned case

→ SELF-SIMILAR SOLUTION



L_b=30 L_c

0<z<50 L_g

δ=2 (out of resonance)

simulation
by G.R.M Robb

$$\left(z_1 = \frac{z - vt}{L_c} \right)$$

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SELF-SIMILAR SUPERRADIANT SOLUTION

$$|A|^2_{peak} \propto z_1^2 \Rightarrow P \propto N^2$$

$$\mathrm{width} \propto z_1^{-1/2} \propto 1/\sqrt{N}$$

particular solution of the FEL eqs:

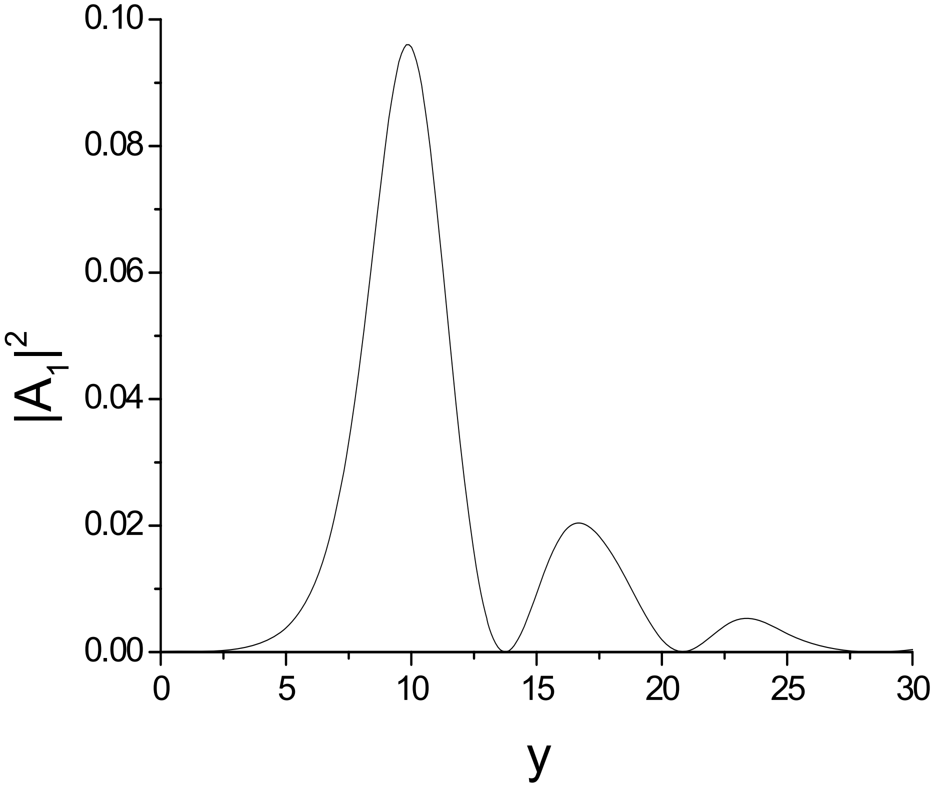
$$A(\bar{z}, z_1) = z_1 A_1(y)$$

$$\theta_j(\bar{z}, z_1) = \theta_{1j}(y)$$

$$y = \sqrt{z_1}(\bar{z} - z_1)$$

$$\frac{d^2\theta_{1j}}{dy^2} = -(A_1 e^{i\theta_{1j}} + c.c.)$$

$$\frac{y}{2}\frac{dA_1}{dy} + A_1 = \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_{1j}}$$



SASE mode for FELs

Ingredients of Self Amplified Spontaneous Emission (SASE)

i) Start up from noise
ii) Propagation effects (slippage)
iii)Superradiance instability

R.Bonifacio, L. De Salvo, P.Pierini, N.Piovella, C. Pellegrini, PRL (1994)

each cooperation length in the e-beam radiates a SR spike which is amplified when it propagates forward on the beam

SASE mode operation to generate 'coherent' X-ray radiation (LCLS, Desy, etc..)

QUANTUM-SASE REGIME

- In a QUANTUM REGIME an FEL behaves as a TWO-LEVEL system
- electrons emit coherent photons as in a LASER
- in the SASE mode the spectrum is intrinsically narrow (‘quantum purification’)
- the transition between the classical and the quantum regime depends on a single parameter:

$$\overline{\rho} = \left(\frac{mc\ \gamma_r}{\hbar k} \right) \rho$$

QUANTUM EFFECTS IN FELs

CLASSICAL LIMIT OF FEL:

electron recoil: $\boxed{mc\,(\delta\gamma\,) >> \hbar k}$: photon recoil

since in the classical regime $\left(\delta\gamma / \gamma_r\right)_{\max} \approx \rho \Rightarrow \boxed{\bar{\rho} >> 1}$ $\quad \bar{\rho} = \left(\frac{mc\,\gamma_r}{\hbar k}\right)\rho$

many recoils implies many photons, hence..
classically, each electron emits many photons

$$\boxed{\frac{< N >_{photon}}{N} = \bar{\rho}\,|\,A\,|^2 >> 1}$$

(since A~1)

the QUANTUM REGIME of FEL occurs when:

$$\boxed{mc\,(\delta\gamma\,) \le \hbar k} \Longrightarrow \boxed{\bar{\rho} < 1}$$

each electron emits only a single photon!

↓

COHERENCE

Quantum FEL behaves like a two-level system
(i.e. a 'laser')

QUANTUM FEL MODEL

Procedure :

Describe N particle system as a Quantum Mechanical ensemble

Write a Schrödinger equation for macroscopic wavefunction $\Psi(\theta, \bar{z})$

or equivalently ..

the equation for the Wigner function (quantum distribution) $W(\theta, p, \bar{z})$

Include slippage (i.e. propagation) using a multiple-scaling approach

$\Psi(\theta, \bar{z}, z_1)$

or

$W(\theta, p, \bar{z}, z_1)$

Canonical Quantization

$$H = \frac{p^2}{2\bar{\rho}} - i\bar{\rho}\left(Ae^{i\theta} - c.c.\right)$$

$$\dot{\theta} = \frac{p}{\bar{\rho}} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\bar{\rho}\left(Ae^{i\theta} + c.c\right) = -\frac{\partial H}{\partial \theta}$$

$$\left[p = \frac{mc(\gamma - \gamma_0)}{\hbar k}\right]$$

$$[\hat{\theta}, \hat{p}] = i \quad \Rightarrow \quad p \rightarrow \hat{p} = -i\frac{\partial}{\partial \theta} \quad \Rightarrow \quad i\frac{\partial \Psi}{\partial \bar{z}} = \hat{H}\Psi$$

$$i\frac{\partial \Psi}{\partial \bar{z}} = -\frac{1}{2\bar{\rho}}\frac{\partial^2 \Psi}{\partial \theta^2} - i\bar{\rho}\left(A(\bar{z})e^{i\theta} - c.c.\right)\Psi$$

$$\frac{dA}{d\bar{z}} = \int_0^{2\pi} d\theta \left|\Psi(\theta, \bar{z})\right|^2 e^{-i\theta} + i\delta A$$

G. Preparata, PRA (1988)

QUANTUM FEL PROPAGATION MODEL

θ describes the spatial evolution of Ψ on the scale of λ

z_1 describes the spatial evolution of A and Ψ on the scale of $L_c >> \lambda$.

$(z_1 = (z - vt)/L_c)$

Using a multiple-scale method we derive the 1D quantum FEL model

$$i\frac{\partial \Psi}{\partial \bar{z}} = -\frac{1}{2\bar{\rho}}\frac{\partial^2 \Psi}{\partial \theta^2} - i\bar{\rho}[A(\bar{z}, z_1)e^{i\theta} - c.c.]\Psi$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \int_0^{2\pi} d\theta \left|\Psi(\theta, \bar{z}, z_1)\right|^2 e^{-i\theta} + i\delta A$$

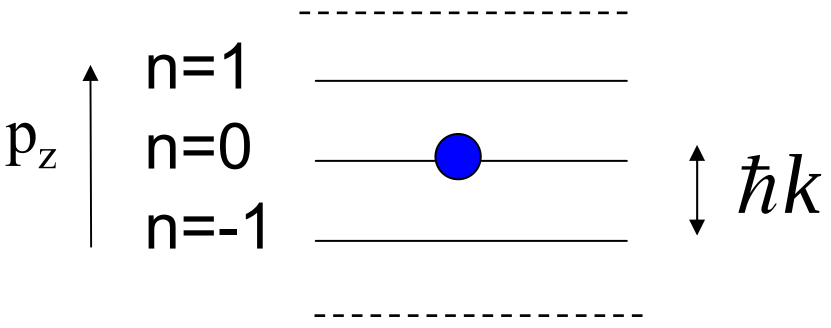
R. Bonifacio, N. Piovella, G.R.M. Robb, M. Cola, Optics Comm. (2005)

Momentum representation:

$$\Psi(\theta,\bar{z},z_1)=\sum_{n=-\infty}^{\infty}c_n(\bar{z},z_1)e^{in\theta}$$

$|c_n|^2$ = probability to find an electron with $p_z = n(\hbar k)$ at z and z_1

discrete values of momentum : $p_z \sim mc\gamma = n\ (\hbar k)$, n=0,±1,..



$$\frac{\partial c_n}{\partial \bar{z}} = -\frac{in^2}{2\bar{\rho}}c_n - \bar{\rho}\left(Ac_{n-1} - A^*c_{n+1}\right)$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \sum_{n=-\infty}^{\infty} c_n c^*_{n-1} + i\delta A$$

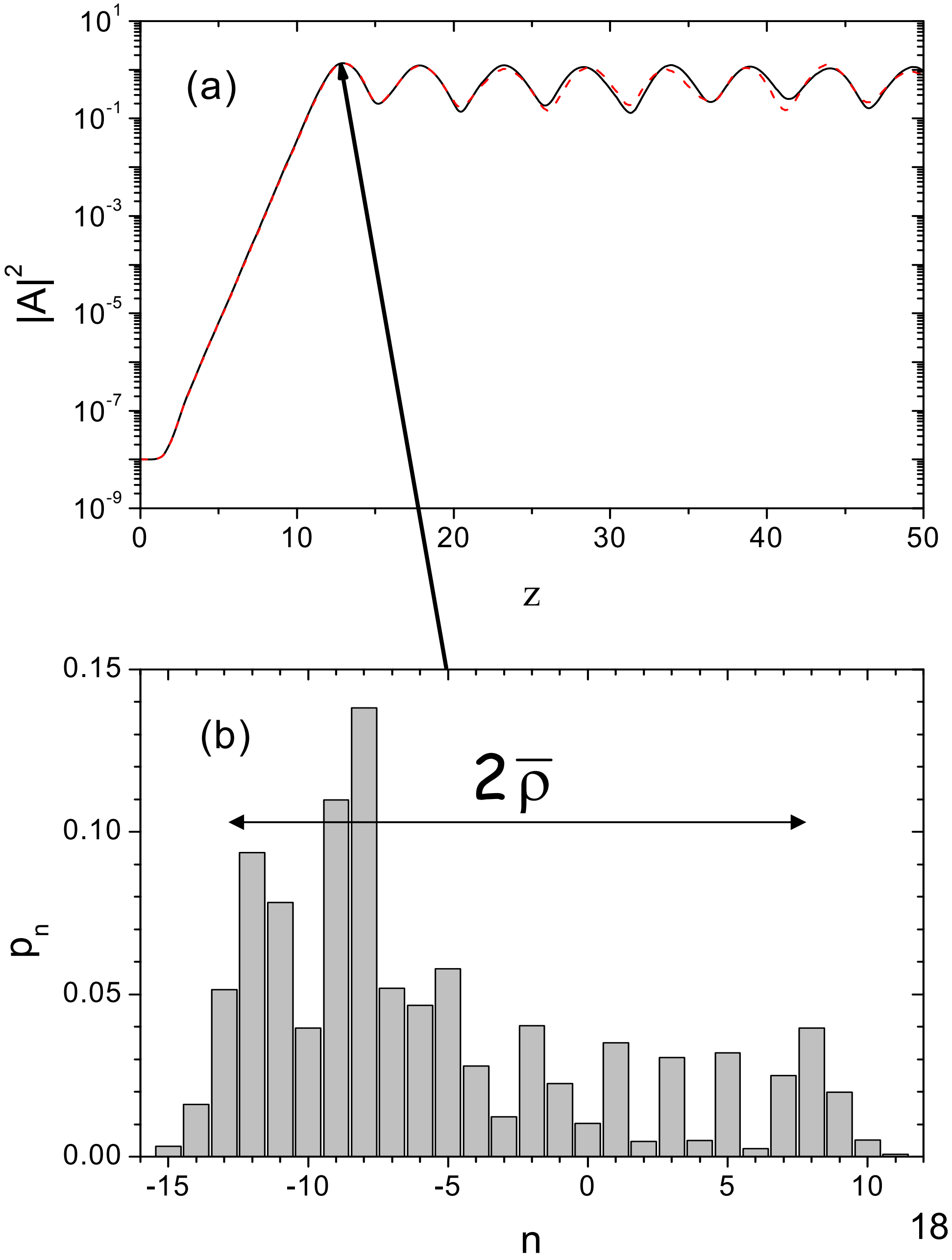
STEADY-STATE regime $\cancel{\left(\frac{\partial A}{\partial z_1}\right)}$

The classical limit is recovered for

$$\bar{\rho} >> 1$$

many momentum states occupied,
both with n > 0 **and** n < 0

$\bar{\rho}$=10, δ=0, no propagation

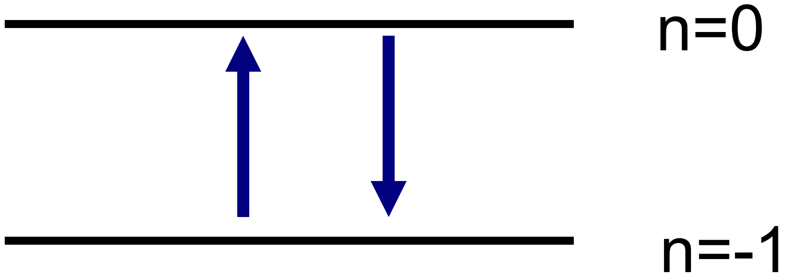


Quantum limit for $\bar{\rho} \leq 1$

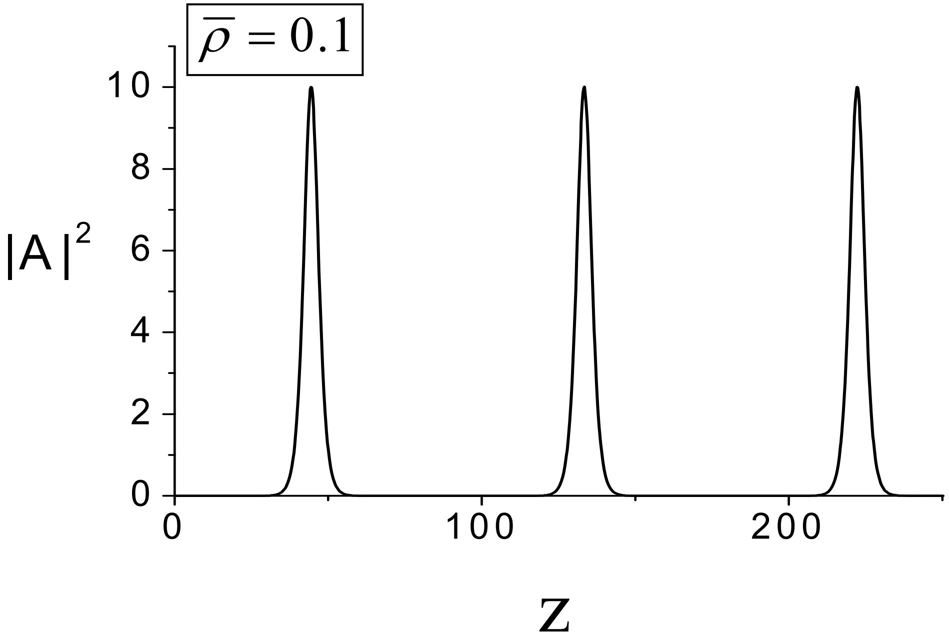
$$\left(\cancel{\frac{\partial A}{\partial z_1}}\right)$$

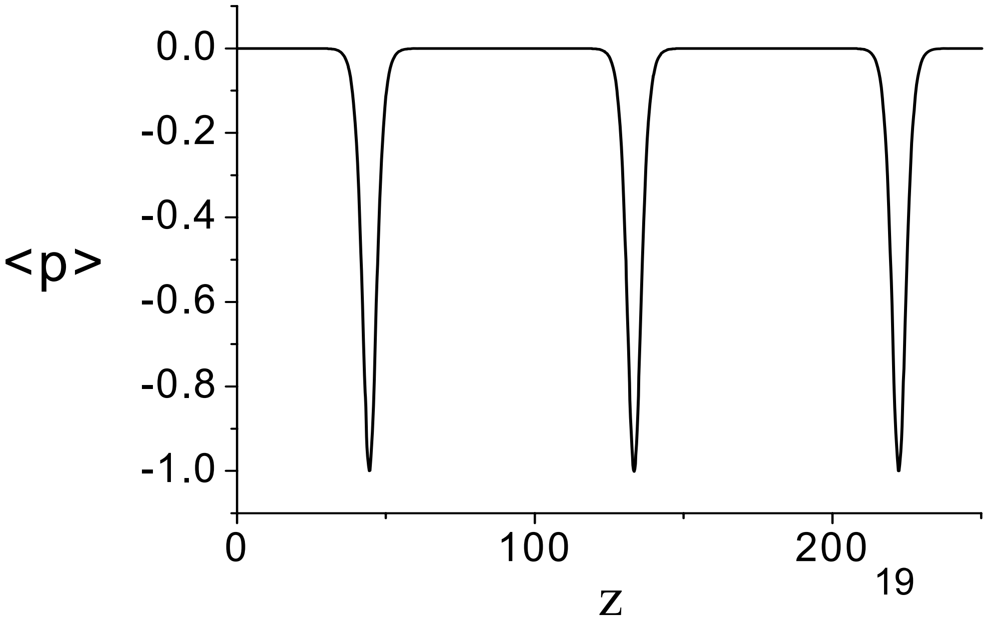
Only TWO momentum states : $p_z = 0$ and $p_z = -\hbar k$

$$\Psi(\theta, \bar{z}) \propto c_0(\bar{z}) + c_{-1}(\bar{z}) e^{-i\theta}$$



Dynamics are those of a 2-level system coupled to an optical field, as in a LASER





Schrödinger vs. Wigner model

Wigner function: quantum phase-space distribution:

$$\Psi(q,t) \to W(q,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dq' e^{-iq'p/\hbar}\Psi^*\left(q-\frac{q'}{2},t\right)\Psi\left(q+\frac{q'}{2},t\right)$$

$\int_{-\infty}^{+\infty} dp W(q,p,t) = |\Psi(q,t)|^2$ space distribution

$\int_{-\infty}^{+\infty} dq W(q,p,t) = |\tilde{\Psi}(p,t)|^2$ momentum distribution

$\iint_{R^2} dq dp W(q,p,t) = 1$ normalization

for a mixed state: $\psi(q) \to \hat{\rho}(t) \to W(q,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dq' e^{-iq'p/\hbar}\left\langle q+\frac{q'}{2}\middle|\hat{\rho}(t)\middle|q-\frac{q'}{2}\right\rangle$

time evolution of the system..

for a pure state:

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}\Psi \quad\Rightarrow\quad \frac{\partial W}{\partial t}=\ldots$$

for a mixed state:

$$i\hbar\frac{d\hat{\rho}}{dt}=\left[\hat{H},\hat{\rho}\right] \quad\Rightarrow\quad \frac{\partial W}{\partial t}=\ldots$$

what about quantum FEL? $\Psi(\theta)$ is a periodic function $(0<\theta<2\pi)$

$\left[\hat{\theta},\hat{p}\right]=i$ analogue to phase and angular momentum operators

→p=m , m=0,±1, …

→W(θ,p) with $p\in(-\infty,+\infty)$ is not well defined!

discrete Wigner function

[J.P. Bizarro, PRA (1994)]

since θ is an angular variable in $(0,2\pi]$, then p is discrete (p=n),

.. we need a discrete Wigner function !

$$W_n(\theta) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta' e^{-2in\theta'} \langle \theta+\theta' | \hat{\rho} | \theta-\theta' \rangle = w_n(\theta) + \sum_{n'=-\infty}^{+\infty} \sin c[(n-n'-1/2)\pi] w_{n'+1/2}(\theta)$$

two 'extra' Wigner functions w_n and $w_{n+1/2}$, with integer and half-integer index:

$$w_s(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta' e^{-2is\theta'} \langle \theta+\theta' | \hat{\rho} | \theta-\theta' \rangle \qquad (s=n \text{ or } s=n+1/2)$$

1D Wigner model for QFEL

N Piovella, MM Cola, L Volpe,
A Schiavi, R Gaiba, R Bonifacio
Optics Comm. (2007)

$$\frac{\partial w_s}{\partial \bar{z}} + \frac{s}{\bar{\rho}}\frac{\partial w_s}{\partial \theta} - \bar{\rho}\left(Ae^{i\theta} + c.c.\right)\left\{w_{s+1/2} - w_{s-1/2}\right\} = 0$$

(s=n or s=n+1/2)

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \sum_{n=-\infty}^{+\infty}\int_{-\pi}^{\pi} d\theta e^{-i\theta} w_{n+1/2}(\theta, z_1, \bar{z}) + i\delta A$$

it is equivalent to the 'c_n' equations for a pure state with:

$$\hat{\rho} = \left|\psi\right\rangle\left\langle\psi\right| \qquad \Psi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$w_{n+\mu/2} = \frac{1}{2\pi}\sum_{n'=-\infty}^{+\infty} e^{i(2n'+\mu)\theta} c^*_{n+n'+\mu} c_{n-n'}$$

(μ=0,1)

CLASSICAL LIMIT:

for $\overline{\rho} >> 1$ $\overline{p} = \frac{s}{\overline{\rho}}$ becomes a continuous variable

,

$$\overline{\rho}\left(w_{s+1/2} - w_{s-1/2}\right) \rightarrow \overline{\rho}\left[w\left(\overline{p} + \frac{1}{2\overline{\rho}}\right) - w\left(\overline{p} - \frac{1}{2\overline{\rho}}\right)\right] \rightarrow \frac{\partial w}{\partial \overline{p}}$$

$$\frac{\partial w}{\partial \overline{z}} + \overline{p}\frac{\partial w}{\partial \theta} - \left(Ae^{i\theta} + c.c.\right)\frac{\partial w}{\partial \overline{p}} = 0$$

$$\frac{\partial A}{\partial \overline{z}} + \frac{\partial A}{\partial z_1} = \int_{-\infty}^{+\infty} d\overline{p} \int_{-\pi}^{\pi} d\theta e^{-i\theta} w(\theta, \overline{p}, z_1, \overline{z}) + i\delta A$$

VLASOV
(or 'Liouville') equation

TO MAKE THE QUANTUM FEL MODEL MORE REALISTIC..
..it must include:

1. initial energy spread (→inhomogeneous broadening (see later..))
2. 3D dynamics (emittance, focusing, optical guiding..)
3. laser wiggler

main points:

A. Quantum effects relevant only longitudinally (photon recoil)
B. Electrons in their transverse motion behave classically
C. Emittance described by a thermal initial distribution..

$\Psi(\theta,x,y)$?? NO, because momentum spread is $\Delta p_x \sim \hbar/(\Delta x)$.

since $\Delta p_x = mc\gamma\Delta x'$, the normalized emittance is

$$\boxed{\varepsilon_n \approx \gamma\,\Delta x\,\Delta x' \approx \frac{\hbar}{mc} = \lambda\!\!\!^{-}_{Compton}}$$

(~ 10^{-7} mm·mrad !!)

SOLUTION:

Electron's 3D dynamics described by the Wigner function

$$W_n(\theta,x,y,p_x,p_y,z,z_1)$$

Radiation 3D evolution described by paraxial Maxwell eqs. for

$$A(x,y,z,z_1)$$

Laser wiggler described by a given (x,y,z) profile

$$a_w(x,y,z)=a_{w0}\ g(x,y,z)$$

From the exact quantum 3D Hamiltonian ..

$$\boxed{\begin{array}{c}\text{from the Liouville} - \text{Von Neuman eq. for a mixed state:}\\ i\hbar\frac{d\hat{\rho}}{dt} = \left[\hat{H}_{3D},\hat{\rho}\right]\end{array}}$$

$$H_{3D} = \underbrace{\frac{p^2}{2\overline{\rho}} - i(g^* A e^{i\theta} - h.c.)}_{\text{longitudinal motion}} + \underbrace{\alpha b\frac{p_\perp^2}{2} + \frac{\xi}{2\alpha\rho X}|g|^2}_{\text{transverse motion}} + \underbrace{\left[\frac{\xi}{2\rho}\left(1 - |g|^2\right) - \alpha^2\frac{b^2}{4a}|\vec{p}_\perp|^2\right]p}_{\text{'mixed' (detuning)}}$$

$$[\theta, p] = i \qquad [x, p_x] = [y, p_y] = i \qquad \left(\vec{p}_\perp = \frac{m\gamma\vec{v}_\perp}{\hbar}, \vec{p}_\perp = -i\vec{\nabla}_\perp\right)$$

(see poster by Mary Cola) 

…to the equation for the 3D Wigner function…

$$w_s(\theta,\vec{x},\vec{p}_\perp) = \frac{1}{2\pi^3}\int d^2x' \int_{-\pi}^{+\pi} d\theta' e^{-2i(s\theta'+\vec{p}_\perp\cdot\vec{x}')}\left\langle \theta+\theta',\vec{x}+\vec{x}'\left|\hat{\rho}\right|\vec{x}-\vec{x}',\theta-\theta'\right\rangle$$

$$(s = m, m+1/2 \qquad m = 0,\pm 1,..)$$

… and to the classical limit for transverse motion

$$\alpha = \frac{\lambda\!\!\!^{-}_{Compton}}{\varepsilon_n} << 1$$

We expand the exact equation for w_s in power of α
and keep the lowest terms

3D QUANTUM FEL MODEL

(Bonifacio, Piovella, Cola, Volpe, Schiavi, *submitted* 2007)

$$\frac{\partial w_s}{\partial \bar{z}} + \left\{ \frac{s}{\bar{\rho}} + \delta + \frac{\xi}{2\rho}\left(1 - |g|^2\right) - \frac{b^2}{4a}\left|p_\perp\right|^2 \right\} \frac{\partial w_s}{\partial \theta} + b(\vec{p}_\perp \cdot \vec{\nabla}_\perp) w_s$$

$$-\bar{\rho}\left(g^* A e^{i\theta} + c.c.\right)\left\{w_{s+1/2} - w_{s-1/2}\right\} - \frac{\xi}{\rho X}\left(\vec{\nabla}_\perp |g|^2\right) \cdot (\vec{\nabla}_{p_\perp} w_s) = 0$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} - ia\nabla_\perp^2 A = g \sum_{m=-\infty}^{+\infty} \int d^2 p_\perp \int_{-\pi}^{\pi} d\theta e^{-i\theta} w_{m+1/2}(\theta, \vec{x}, \vec{p}_\perp, \bar{z}, z_1)$$

s=0, ± 1/2, ± 1,..

transverse variables:

$\vec{x}$ in units of σ (spatial spread Δx)

$\vec{p}_\perp$ in units of $\varepsilon_n/\gamma\sigma$ ('momentum' spread $\Delta x'$)

3D parameters:

$$\boxed{b = \frac{L_g}{\beta^*} \quad , \quad \beta^* = \frac{\gamma\sigma^2}{\varepsilon_n} \text{ ('beta function ')}}$$

$$\boxed{a = \frac{L_g}{Z_r} \quad , \quad Z_r = \frac{4\pi\sigma^2}{\lambda_r} \text{ (radiation Rayleigh range)}}$$

$$\boxed{\xi = \frac{a_{w0}^2}{1 + a_{w0}^2}}$$

(only for laser wiggler)

$$\boxed{X = \frac{b}{a} = \frac{Z_r}{\beta^*} = \frac{4\pi\varepsilon_n}{\gamma\lambda_r}}$$

interpretation:

$$\boxed{\begin{array}{l}\left\{\cdots\frac{\xi}{2\rho}\left(1-|g|^2\right)\cdots\right\}p\frac{\partial w_s}{\partial\theta}\\ \left\{\cdots\frac{b^2}{4a}\left|\vec{p}_\perp\right|^2\cdots\right\}p\frac{\partial w_s}{\partial\theta}\end{array}}$$

→ detuning from laser wiggler ($\Delta a_w/a_w$)

→ resonance spread from emittance ($\gamma^2\Delta\theta_{x,y}{}^2$)

$$\boxed{\ldots b\left(\vec{p}_\perp\cdot\vec{\nabla}_\perp w_s\right)\ldots}$$

→ beam expansion due to emittance

$$\boxed{\ldots-\frac{\xi}{2\rho X}\left(\vec{\nabla}_\perp|g|^2\right)\cdot\left(\vec{\nabla}_{p_\perp}w_s\right)\ldots}$$

→ laser wiggler gradient force

3D CLASSICAL LIMIT: from the Wigner to the Vlasov equation..

$$\boxed{\bar{\rho} >> 1}$$

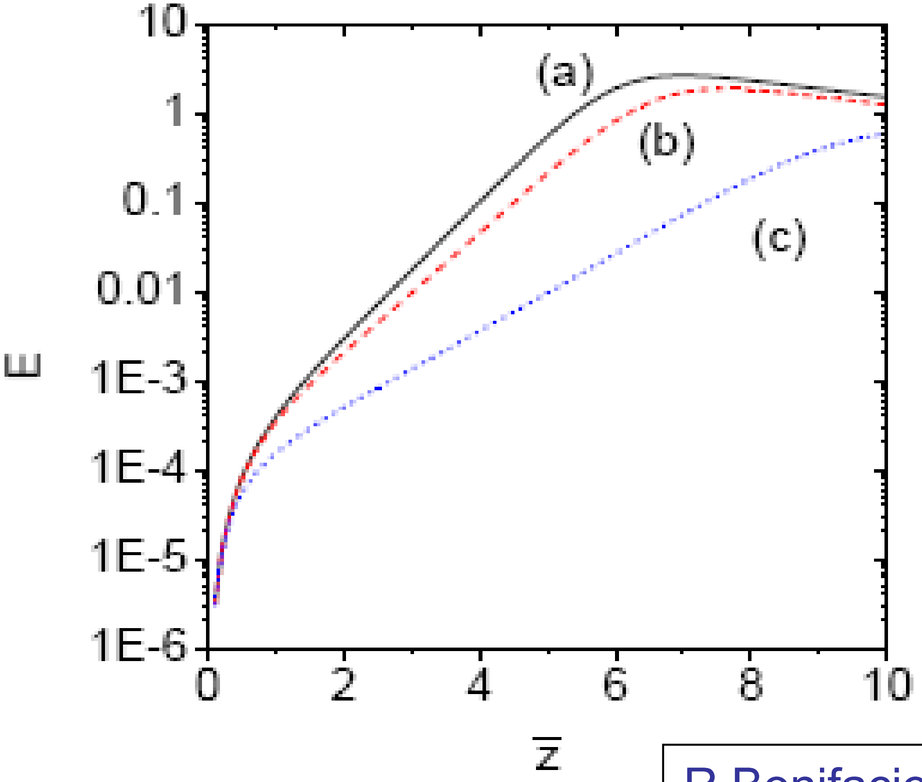
$w_s(\theta,x,y,p_x,p_y,z,z_1) \rightarrow W(\theta,p,x,y,p_x,p_y,z,z_1)$ with $p = s/\bar{\rho}$ continuous

$$\boxed{\begin{aligned}
&\frac{\partial W}{\partial \bar{z}} + \left\{ p + \delta + \frac{\xi}{2\rho}\left(1 - |g|^2\right) - \frac{b^2}{4a}\left|\vec{p}_\perp\right|^2 \right\} \frac{\partial W}{\partial \theta} + b\vec{p}_\perp \cdot \vec{\nabla}_\perp W \\
&- \left(Ae^{i\theta} + c.c.\right)\frac{\partial W}{\partial p} - \frac{\xi}{2\rho X}\left(\nabla_{\bar{x}} |g|^2\right) \cdot \vec{\nabla}_{p_\perp} W = 0 \\
&\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} - ia\nabla_\perp^2 A = g\int d^2p_\perp \int_{-\infty}^{+\infty} dp \int_{-\pi}^{\pi} d\theta e^{-i\theta} W(\theta, p, \vec{x}, \vec{p}_\perp, z_1, \bar{z})
\end{aligned}}$$

NUMERICAL RESULTS: (see Angelo Schiavi's next talk..)
example: inhomogenous emittance

[(a) ε_n=0, (b) ε_n=0.05 mm mrad, (c) ε_n=0.1 mm mrad]

[for instance λ_r=2 Å, λ_L=1μm, γ=36, I=800 A, σ=10μm, a_{w0}=0.15, L_g=1.3 mm]



$$\frac{b}{2\sqrt{a}} = 0.5\,(b)\,,\,1\,(c)$$

$$Z_r \approx 6250 L_g \qquad (a = 1.6 \cdot 10^{-6})$$

$$\boxed{\frac{b}{2\sqrt{a}} < 1 \Rightarrow \varepsilon_n < \frac{\gamma\lambda_r}{2\pi}\sqrt{\frac{Z_r}{L_g}}}$$

R Bonifacio, N Piovella, MM Cola, L Volpe, NIMA 577 (2007)

CONCLUSIONS

we have presented a 3D Quantum FEL model, based on a Wigner function for the electron beam.

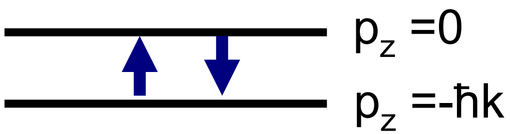
The model include:

- Optical diffraction (guiding)
- Transverse effects in the electron beam (defocusing, emittance ..)
- Laser wiggler

- The equations couple the longitudinal and the transverse dynamics.
- The longitudinal recoil is quantized in units of $\hbar k$.
- The electron beam behaves classically in its transverse motion.

QUANTUM REGIME (TWO-LEVEL APPROXIMATION)

$$\boxed{\bar{\rho} < 1}$$



only two momentum states p_z =0 and p_z =-ħk

$$D = \int_{-\pi}^{+\pi} d\theta \left[w_0(\theta) - w_{-1}(\theta) \right]$$ population difference

$$B = \int_{-\pi}^{+\pi} d\theta\, w_{-1/2}(\theta) e^{-i\theta}$$ bunching ('polarization')

'quasi-Bloch' equations
(similar as for lasers)

$$\frac{\partial D}{\partial \bar{z}} + b\vec{p}_\perp \cdot \nabla_\perp D = -2\left(g^* A B^* + c.c\right) + \upsilon\left(\nabla_\perp |g|^2\right) \cdot \nabla_{p_\perp} D$$

$$\frac{\partial B}{\partial \bar{z}} + b\vec{p}_\perp \cdot \nabla_\perp B = i\left\{\Delta - \bar{\xi}\left(1 - |g|^2\right) - \frac{b^2}{4a}\left|\vec{p}_\perp\right|^2\right\} B + g^* D A + \upsilon\left(\nabla_\perp |g|^2\right) \cdot \nabla_{p_\perp} D$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} - ia\nabla_\perp^2 A = g \int d^2 p_\perp B$$

$$L_g \rightarrow L_g / \sqrt{\bar{\rho}} \;,\; A \rightarrow \sqrt{\bar{\rho}} A \qquad \bar{\xi} = \frac{\xi}{2\rho\sqrt{\bar{\rho}}} \;,\; \upsilon = \frac{\xi}{\rho\sqrt{\bar{\rho}} X} \;,\; \Delta = \frac{mc(\gamma_0 - \gamma_R) - \hbar k/2}{\hbar k \bar{\rho}^{3/2}}$$

How to model the INITIAL ENERGY SPREAD?

$$\frac{\partial c_n}{\partial \bar{z}} = -in\left(\frac{n}{2\bar{\rho}} + \delta\right)c_n - \bar{\rho}\left(Ac_{n-1} - A^* c_{n+1}\right)$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \sum_{n=-\infty}^{\infty} \int d\delta\, G(\delta)\, c_n c^*_{n-1}$$

G(δ) ; initial distribution of width σ

$$A \rightarrow A e^{i\delta\bar{z}}$$

$$c_n \rightarrow c_n e^{in\delta\bar{z}}$$

linear theory → GENERALIZED DISPERSION RELATION :

$$\left(A \propto e^{i\lambda\bar{z}}\right)$$

$$\lambda - \Delta + \bar{\rho}\int_{-\infty}^{\infty} \frac{d\delta}{\lambda+\delta}\left\{G\left(\delta + \frac{1}{2\bar{\rho}}\right) - G\left(\delta - \frac{1}{2\bar{\rho}}\right)\right\} = 0$$

$$\left(\Delta = \delta + \frac{n}{\bar{\rho}} - \bar{\omega}\right)$$

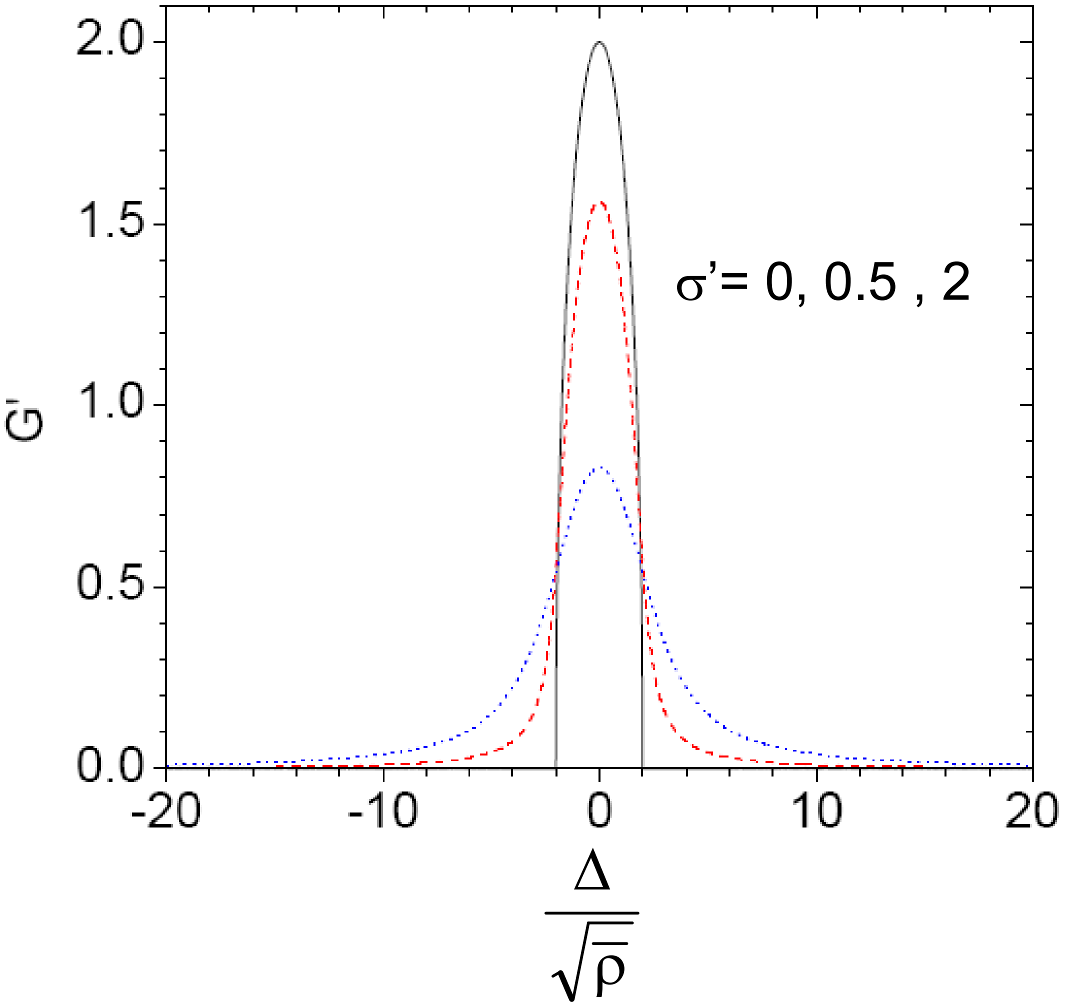
N. Piovella, R. Bonifacio, NIMA (2006)

when $G(\delta)$ is Lorentzian → $\boxed{(\lambda-\Delta)\left[(\lambda-i\sigma)^2-\frac{1}{4\bar{\rho}^2}\right]+1=0}$

for $\bar{\rho}<1$

$$\boxed{\text{max gain}=\frac{1}{L_g}\left[\sqrt{4+\sigma'^2}-\sigma'\right]}$$

$$\sigma'=\frac{1}{\rho\sqrt{\bar{\rho}}}\left(\frac{\delta\gamma}{\gamma}\right)_0$$



$\overline{\rho} = 0.1$

$\left(\frac{\delta\gamma}{\gamma}\right)_0 = 0.1\ \rho$

population of n=0
(p=-ħk)

population of n=0
(p=0)

$\overline{\rho} = 0.1$

$$\left(\frac{\delta\gamma}{\gamma}\right)_0 = \rho$$

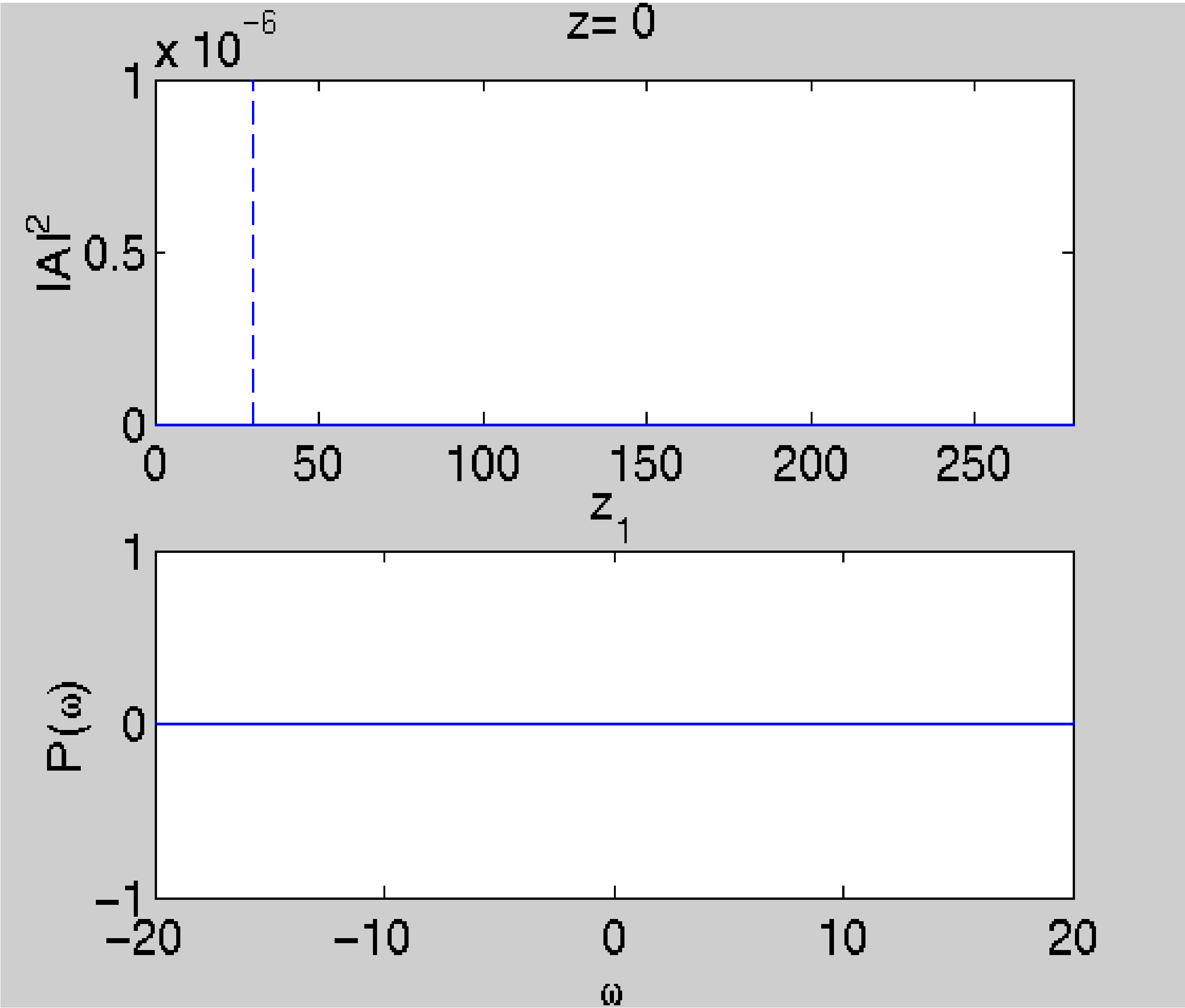
population of n=0
(p=-ħk)

population of n=0
(p=0)

$\bar{\rho} = 0.1$

SASE with energy spread

$\left(\frac{\delta\gamma}{\gamma}\right)_0 = 0.5\rho$



coherence is preserved if

$$\boxed{mc\delta\gamma_0 < \hbar k}$$

but gain decreases for

$$\boxed{\left(\frac{\delta\gamma}{\gamma}\right)_0 > \rho\sqrt{\bar{\rho}}}$$

simulation by G.R.M Robb