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Introduction to Synchrotron Radiation and Free Electron Lasers

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Outlook

- FROM UNDULATOR TO FEL
- 1D AND 3D EQUATIONS
- NUMERICAL RESULT
- CONCLUSION

PHASE SPACE DESCRIPTION

$$\sigma_x(z) = \sqrt{\epsilon_x \left(\beta_x^* + \frac{s^2}{\beta_x^*}\right)}, \quad \beta^* = \sigma_x/\sigma_x' \quad \sigma_x\sigma_x' = \epsilon_x$$ electron beam size

$$\sigma_r(z) = \sqrt{\epsilon_r \left(Z_r + \frac{s^2}{Z_r}\right)}, \quad Z_r = \sigma_r/\sigma_{r'} \quad \sigma_r\sigma_r' = \epsilon_r = \frac{\lambda}{4\pi}$$ radiation beam size

ELECTRON-BEAM + RADIATION BEAM

$$\Sigma_x = \sqrt{\sigma_x^2 + \sigma_r^2}, \quad \Sigma_{x'} = \sqrt{\sigma_{x'}^2 + \sigma_{r'}^2}$$

$\Sigma_x\Sigma_{x'} \simeq \frac{\lambda}{4\pi},$ radiation beam is choerent

$\Sigma_x\Sigma_{x'} \gg \frac{\lambda}{4\pi},$ radiation beam inchoerent

$M_t = \frac{\Sigma_x\Sigma_{x'}}{\frac{\lambda}{4\pi}} = \frac{\epsilon_x}{\epsilon_r},$ Number of coherent tranverse mode

If $M_t \simeq 1 \rightarrow \epsilon_x \simeq \epsilon_r$ If $M_t \gg 1 \rightarrow \epsilon_x \gg \epsilon_r$

BRIGHTNESS I

$\lambda_1(\psi) = \frac{\lambda_w}{2\gamma^2}(1 + K^2/2 + \gamma^2\psi^2)$ resonant wavelength

$\frac{\Delta\omega_1}{\omega_1} = \simeq \frac{1}{N_w} \rightarrow \Delta\psi \simeq \sqrt{\lambda_1/L_w} \quad \Delta x = \sqrt{\lambda_1 L_w} \rightarrow \Delta\psi\Delta x \simeq \lambda_1$

$\langle \frac{dn}{(d\omega/\omega_1)d\Omega} \rangle = \alpha N_w^2 \gamma^2 [KK]^2$ n of ph in a bandwidth, in solid angle $\Delta\psi^2 \simeq 1/(\gamma^2 N_w)$

$B = \frac{dn/dt}{A_t} = \frac{\text{spectral flux}}{\text{tranverse phase space area}}$ brightness

Spectral flux= number of photons per unit time in a given bandwidth $\frac{\Delta\omega_1}{\omega_1} \ll 1/N_w$

$\frac{dn}{dt} = \pi\alpha N_w (I/e)(\Delta\omega_1/\omega_1)[KK]^2$

$A_t = (4\pi)^2 \Sigma_{x_t} \Sigma_{x'_t}$

BRIGHTNESS II

In a tipical third generation light source we have:

$$B = \frac{\pi \alpha N_w (I/e)(\Delta\omega_1/\omega_1)[KK]^2}{(4\pi)^2 \Sigma_{x_t} \Sigma_{x'_t}}$$

$$\Sigma_x \Sigma_{x'} \simeq \epsilon_x \simeq 10^{-2} mm - mrad, \quad \Sigma_y \Sigma_{y'} \simeq \epsilon_y \simeq 10^{-4} mm - mrad,$$
$$(I/e) \simeq 100 mA/e \simeq 10^{18}/sec, \quad \alpha N_w \simeq 1, \quad \Delta\omega/\omega = 10^{-3}$$

Then we have:

$$B = \simeq 10^{20} \left(\frac{\text{ph per second in } 0.1\% \text{ BW}}{(\text{mm})^2 (\text{mrad})^2} \right)$$

Undulator in a Storage Ring:

$$N_w = 100 \quad \lambda_w = 5cm \quad K = 1 \quad \epsilon_x = 4 \times 10^{-9} m - rad$$

$$\epsilon_y = 4 \times 10^{-11} m - rad \quad \beta^* = 5m \quad I = 100mA$$

$$E_{ph} = \hbar\omega_1 = \frac{\hbar c}{\lambda_1(\psi)} \simeq \frac{\hbar c}{\lambda_1(0)} \rightarrow \lambda_1 = \frac{\hbar c}{E_{ph}}$$

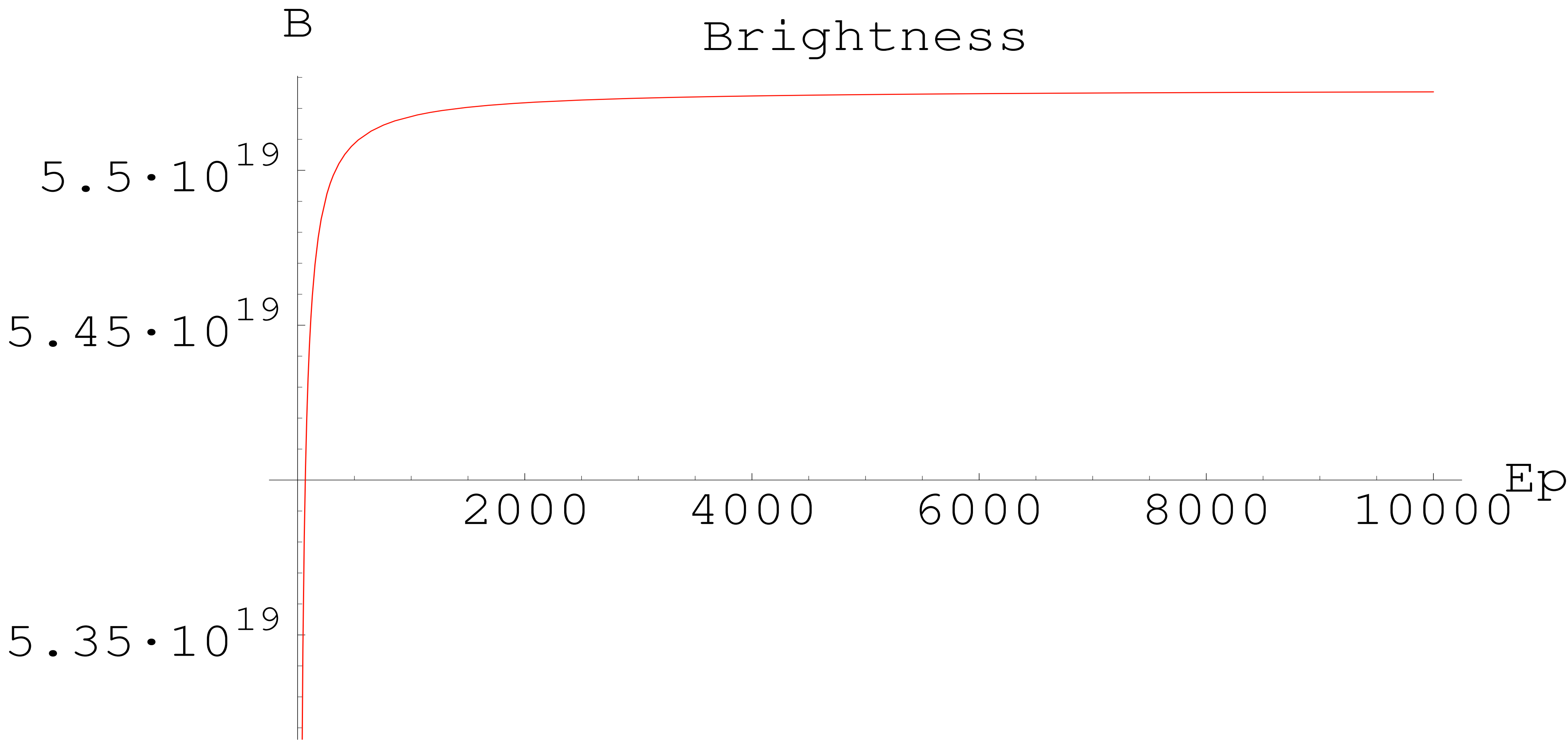
$$\Sigma_x = \sqrt{\beta^* \epsilon_x + \frac{\hbar c L_w}{8\pi^2} \frac{1}{E_{ph}}} \quad \Sigma_{x'} = \sqrt{\epsilon_x / \beta^* + \frac{\hbar c}{L_w \pi} \frac{1}{E_{ph}}}$$

$$B = B(E_{ph}) \quad M_x(E_{ph}) = \frac{4\pi}{\hbar c} \epsilon_x E_{ph} \qquad 10eV \lesssim E_{ph} \lesssim 10^4 eV$$

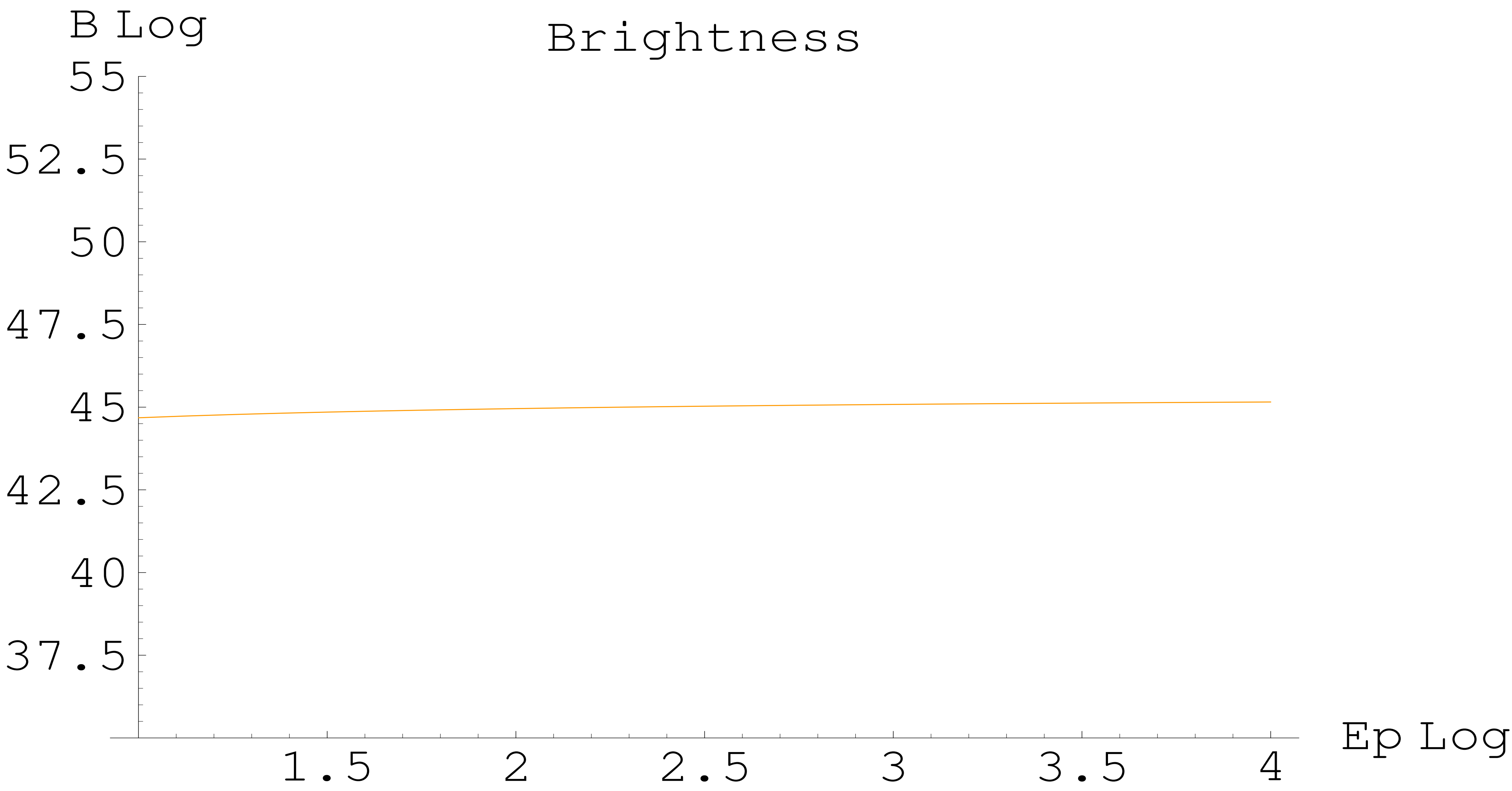
$$\lambda_1(10) = 48\mathring{A} \quad M_x(10) \simeq 10 \quad M_y(10) \simeq 10^{-1}$$

$$\lambda_1(10^4) = 4,8 \times 10^{-2}\mathring{A} \quad M_x(10^4) \simeq 10^4 \quad M_y(10^4) \simeq 10^2$$

BRIGHTNESS AND COHERENCE II



BRIGHTNESS AND COHERENCE III



FROM UNDULATOR TO FEL

- Tranverse enhancement:
  The FEL radiation is complętly coherent $A_t \simeq \epsilon_x \epsilon_y \rightarrow (\lambda/4\pi)^2$
  If $\lambda = 1.5\mathring{A}$, give an anachement factor $\frac{\epsilon_x \epsilon_y}{(\lambda/4\pi)^2} \simeq 10^4$
- Temporal enhancement:
  In a FEL the electron in one coerence length radiate together, the intensity enhancement is
  Number of electrons in one L_c $\quad N_{l_c} \simeq \frac{I}{ec}\frac{\lambda}{\rho} \simeq 10^6$
- The bunch length being squeezed by bunch compressor to 100 femtosecond $l_b \simeq tc = 3cm$ $\quad$ (for $t \simeq ps \quad l_b \simeq 30cm$)

EXAMPLE Brightness in LCLS: $B \simeq 10^{33}$ fully coherence hypothesis

1D PARTICLE MODEL

$$\partial_{\bar{z}}\theta = \bar{\eta} \qquad \partial_{\bar{z}}\bar{\eta} = \tilde{a}e^{i\theta} + c.c$$

$$\left(\partial_{\bar{z}} + \frac{1}{2\rho}\partial_\theta + \frac{1}{4ik_w k\rho}\nabla_\perp^2\right)\tilde{a} = \langle e^{-i\theta}\rangle_\Delta$$

$$\boxed{\theta = (k + k_w)(z - \bar{v}_z t) \quad \bar{z} = z/\sqrt{3}L_g \quad \bar{\eta} = (\gamma - \gamma_r)/\rho\gamma_r}$$

$$\boxed{\tilde{a} = \frac{\kappa_1}{k_w\rho^2}\tilde{E} \rightarrow \rho|\tilde{a}|^2 \simeq \rho \simeq \frac{n_e mc^2\gamma_0}{2\epsilon_0|E|^2} = \frac{\text{Field Energy}}{\text{e}-\text{beam Energy}}}$$

$$\rho = \left[\frac{1}{8\pi}\frac{I}{I_A}\left(\frac{K[JJ]}{1+K^2/2}\right)\frac{\gamma\lambda_1^2}{\Sigma_A}\right]^{1/3}$$

$$L_g = \frac{\lambda_w}{4\pi\sqrt{3}\rho} \quad \lambda_1 = \frac{\lambda_w}{2\gamma^2}\left(1 + K^2/2 + \gamma^2\psi^2\right) \quad K = \frac{eB_0}{mck_w}$$

$$\kappa_1 = eK[JJ]/(4\gamma_0^2 mc^2) \quad \kappa_2 = eK[JJ]/(2\epsilon_0\gamma_0)$$

LINEAR APROXIMATION

$$\partial_{\bar{z}}\tilde{a} = -b \qquad \partial_{\bar{z}}b = -iP \qquad \partial_{\bar{z}}P = \tilde{a}$$

$b = \langle e^{-i\theta}\rangle_\Delta$ Bunching parameter $P = \langle \bar{\eta}_j e^{-i\theta}\rangle_\Delta$ collettive momentum

$$\frac{d^3\tilde{a}}{d\bar{z}^3} = i\tilde{a} \qquad \tilde{a} \propto e^{-i\mu\bar{z}} \rightarrow \mu^3 = 1$$

$$\tilde{a}(\bar{z}) = \frac{1}{3}\sum_{l=1}^{3}\left\{\tilde{a}(0) - i\frac{b(0)}{\mu_l} - iP(0)\mu_l\right\} e^{-i\mu_l\bar{z}}$$

SASE: $\tilde{a}(0) = 0 \;\; P(0) = 0$ $\quad \langle\tilde{a}\tilde{a}^*\rangle \simeq \langle|b(0)|^2\rangle e^{\sqrt{3}\bar{z}} \;\; \langle|b(0)|^2\rangle \simeq \frac{1}{N_{l_c}}$

N_{l_c}= number of electrons in a coherent length $l_c \simeq \lambda/\rho$

1D VLASOV EQUATIONS

Klimontovich Distribution

$$F(\theta,\eta;z) = \frac{k_1}{(dN_e/dz)} \sum_{j=1}^{N_e} \delta(\theta-\theta_j)\delta(\eta-\eta_j)$$

$$\left(\partial_z + \dot{\theta}\partial_\theta + \dot{\eta}\partial_\eta\right) F(\theta,\eta;z) = 0$$

Frequency Domain

$$E(z,t) = \frac{1}{2}\int d\nu E_\nu(z) e^{i\Delta\nu k_1(z-ct)} + c.c.$$

$$F_\nu(\eta;z) = \frac{1}{2\pi}\int d\theta e^{-i\nu\theta} F(\theta,\eta;z)$$

LINEAR APPROXIMATION I

$$F(\theta, \eta; z) = V(\eta) + \Delta F(\theta, \eta; z)$$

$$\left(\partial_z + i\Delta\nu k_w\right) E_\nu(z) = -\kappa_2 n_0 \int d\eta F_\nu(\eta; z)$$

$$\left(\partial_z + i\nu 2k_w\eta\right) F_\nu(\eta; z) + \kappa_1 E_\nu(z) d_\eta V(\eta) = 0$$

$$\dot{\theta} = 2k_w\eta \qquad \dot{\eta} = \kappa_1 \int d\nu e^{i\nu\theta} E_\nu + c.c.$$

$$\Delta\nu = 1 - \nu \quad n_0 = dN_e/(dz\Sigma_A)$$

LINEAR APPROXIMATION II

$$E_\nu(z) = \oint \frac{d\mu}{2\pi i} \frac{e^{i\mu\bar{z}}}{D(\mu)} \left[E_\nu(0) + \frac{i\kappa_2 n_0}{2\rho k_w N_\lambda} \sum_{j=1}^{N_e} \frac{e^{-\nu\theta_j(0)}}{\eta_j(0)/\rho - \mu} \right]$$

$$D(\mu) = \mu - \frac{\Delta\nu}{2\rho} - \int d\eta \frac{V(\eta)}{(\eta/\rho - \mu)^2}$$

The Dynamics of the system is determinate by the the singularity a
the solution of the dispersion relation
For e-beam with vanishing energy spread

$$V(\eta) = \delta(\eta) \rightarrow D(\mu) = \mu^3 - 1$$

FROM 1D TO 3D

- DIFFRACTION EFFECT
- EMITTANCE OR ANGULAR SPREAD EFFECT
- ENERGY SPREAD EFFECT

Qualitative discussion

Decreasing the average beta function (focusing whit quadrupole) will actually degrade the fel performance, because the large angular spread introduces a spread in a resonant wavelength

$$\frac{\Delta\lambda_1}{\lambda_1} = {\sigma'}_x^2 \frac{\lambda_w}{\lambda_1} \simeq \rho \rightarrow \sigma'_x \lesssim \sqrt{\frac{\epsilon_r}{L_g}} \simeq \sigma'_r$$

the angular divergence should be no more than the angular divergence of radiation.

for a given emittance of the e-beam the beta function should be:

$$\beta_x \simeq L_g \frac{\epsilon_x}{\epsilon_r}$$

a smaller beam emittance allow a tighter focused beam size

3D EQUATIONS I

$$\partial_{\bar z}\theta = \bar\eta - \frac{1}{2}\left(\bar p^2 + 3(\bar k_\beta L_{g_0})^2 \bar x^2\right) \qquad \partial_{\bar z}\bar\eta = \int d\nu \tilde a_\nu e^{i\theta} + c.c$$

$$d_{\bar z}\bar p = (3L_{g_0})^{(2/3)} \bar k_\beta^2 \bar x \qquad d_{\bar z}\bar x = \bar p / \sqrt{3L_{g_0}}$$

$$\boxed{\bar x = x\sqrt{k_1/\sqrt{3}L_{g_0}} \quad \bar p = p\sqrt{\sqrt{3}k_1 L_{g_0}}}$$

$$\left(\partial_{\bar z} + i\bar\nu + \frac{1}{2i}\nabla_\perp^2\right)\tilde a(\bar x, \bar z) = \frac{\rho}{\pi}\int d\theta \int d^2\bar p \int d\bar\eta f(\theta, \bar\eta, \bar x, \bar p; \bar z)$$

$$\left(\partial_{\bar z} + \dot\theta\partial_\theta + \bar p\partial_{\bar x} - \bar k_\beta^2 \bar x \partial_{\bar p} + \int d\bar\nu e^{i\bar\nu\theta}\partial_{\bar\eta}\right) f(\theta, \bar\eta, \bar x, \bar p; \bar z) = 0$$

$$\boxed{\bar\nu = \Delta\nu/2\rho \quad a_\nu = 2\sqrt{3}k_1 L_{g_0} E_\nu e^{-i\bar\nu\bar z} \quad f = \rho/\sqrt{3}k_1 L_{g_0}}$$

3D DISPERSION RELATION

$$f_0(\bar{p}^2 + \bar{k}_\beta^2, \eta) = \frac{1}{2\pi\bar{\sigma}_x^2\bar{k}_\beta^2} e^{-\frac{\bar{p}^2+\bar{k}_\beta^2\bar{x}^2}{2\bar{\sigma}_x^2\bar{k}_\beta^2}} \frac{1}{\sqrt{2\pi\bar{\sigma}_\eta}} e^{-\frac{\bar{\eta}^2}{2\bar{\sigma}_\eta}}$$

$$\boxed{\bar{\sigma}_x = \sigma_x\sqrt{2k_1 k_w \rho} \quad \bar{\sigma}_\eta = \sigma_\eta/\rho \quad \epsilon = \bar{\sigma}_x^2\bar{k}_\beta/k_1}$$

scaled rms tranverse beam size, rms energy spread and the electtron beam emittance

Performing Van Kampen's Normal Mode Expansion we obtain the 3D dispersion relation:

$$\left(\mu - \bar{\nu} + \frac{1}{2}\nabla_\perp^2\right) A(\bar{x}) = \frac{1}{2\pi\bar{k}_\beta^2\bar{\sigma}_x^2} \int_{-\infty}^{0} \tau d\tau e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}$$

$$\int d^2\bar{p}\, A\left(\bar{x}\cos(\bar{k}_\beta\tau) + \frac{\bar{p}}{\bar{k}_\beta}\sin(\bar{k}_\beta\tau)\right) e^{-\frac{\bar{p}^2+\bar{k}_\beta^2\bar{x}^2}{2}} \left(i\tau + \frac{1}{\bar{k}_\beta^2\bar{\sigma}_x^2}\right)$$

3D TRANVERSE EFFECT

- Diffraction effect

$$\bar{\sigma}_x^2 = \sigma_x^2 2k_1 k_w \rho = \frac{2}{\sqrt{3}} \frac{Z_r}{L_{g0}} \qquad Z_r > L_{g0}$$

- Emittance or angular spread effect

  - $\bar{\sigma}_x^2 \bar{k}_\beta = \frac{\epsilon}{2\epsilon_r}$ $\epsilon \simeq \epsilon_r$ in order to generate coherent radiation
  - $(\bar{\sigma}_x \bar{k}_\beta)^2 = \sigma_x'^2 \frac{\lambda_w}{\lambda_1} 2k_u^2 \rho = \frac{\Delta\lambda_1}{\lambda_1} 4\pi\sqrt{3} \frac{L_{g0}}{\lambda_w} = 4\pi\sqrt{3} \frac{\Delta\lambda_1}{\lambda_1} N_{g0}$ the beam

  angular spread $\sigma_x' = k_\beta \sigma_x$ introduces a spread in a resonant wavelength, the resonant condition can be mantained for the whole beam if $\bar{\sigma}_x^2 \bar{k}_\beta < 1$

- Energy spread effect

$$\bar{\sigma}_\eta = \frac{\Delta\gamma}{\gamma\rho} \simeq \frac{\Delta\lambda_1}{\lambda_1 \rho} = 4\pi\sqrt{3} \frac{\Delta\lambda_1}{\lambda_1} N_{g0} \lesssim 1$$

the resonant wavelength spread caused by energy spread must be less then unit

FITTING FORMULA I

From a numerical solutions of the dispersion relation we can built a very useful fitting formula.

$$L_g = L_{g_0}(1+\Lambda)$$

$$E = P_{sat}/P_{beam} = \rho\frac{1.6}{(1+\Lambda)^2}$$

$$\Lambda = \Lambda(\alpha_i, \eta_j) \quad i = 1, 19 \quad j = 1, 3$$

$$\eta_1 = \frac{L_{g0}}{2k_1\sigma_x^2} \qquad \eta_2 = 4\pi\frac{L_{g0}}{\lambda_\beta}k_1\epsilon \qquad \eta_3 = 4\pi\frac{L_{g0}}{\lambda_w}\sigma_\eta$$

where η_j are respectively diffraction, angular spread and energy spread parameter

FITTING FORMULA II

$$\rho = R\beta^{-1/3} \qquad L_{g_0} = \frac{\lambda_w}{4\sqrt{3}\pi R}\beta^{1/3}$$

$$R = \left[\frac{1}{(4\pi)^2}\frac{I}{I_A}\left(\frac{K[JJ]}{1+K^2/2}\right)\frac{\gamma\lambda_1^2}{\epsilon_x}\right]^{1/3}$$

$$\eta_1 = \frac{L_{g_0}}{2k_1\sigma_x^2} = \left(\frac{\lambda_w\lambda_1}{(4\pi)^2\sqrt{3}\epsilon_x R}\right)\beta^{-2/3}$$

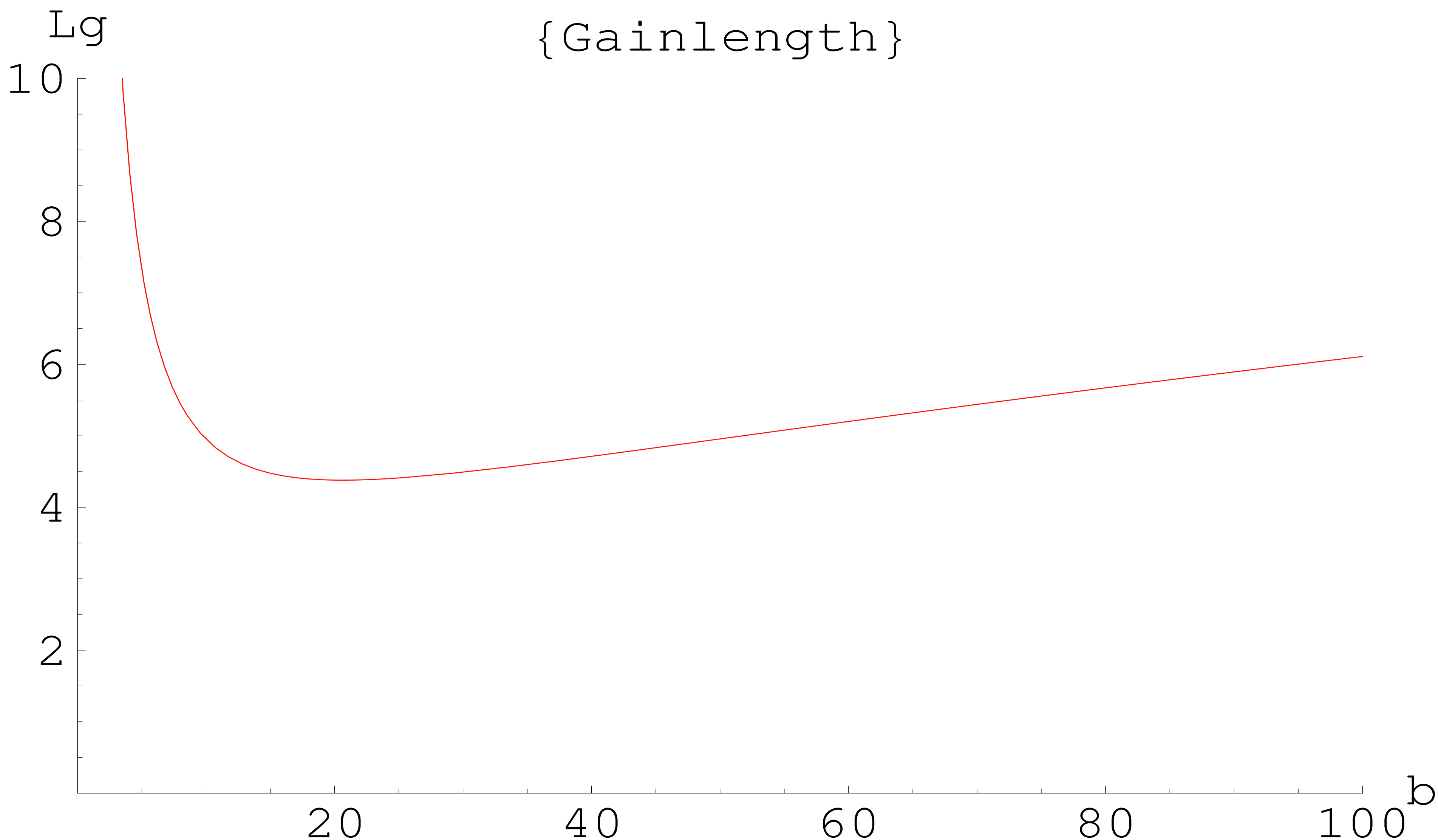
$$\eta_2 = 4\pi\frac{L_{g_0}}{\lambda_\beta}k_1\epsilon = \left(\frac{\lambda_w\epsilon_x}{\sqrt{3}R\lambda_1}\right)\beta^{-2/3}$$

$$\eta_3 = 4\pi\frac{L_{g_0}}{\lambda_w}\sigma_\eta = \left(\frac{\sigma_\eta}{\sqrt{3}R}\right)\beta^{1/3}$$

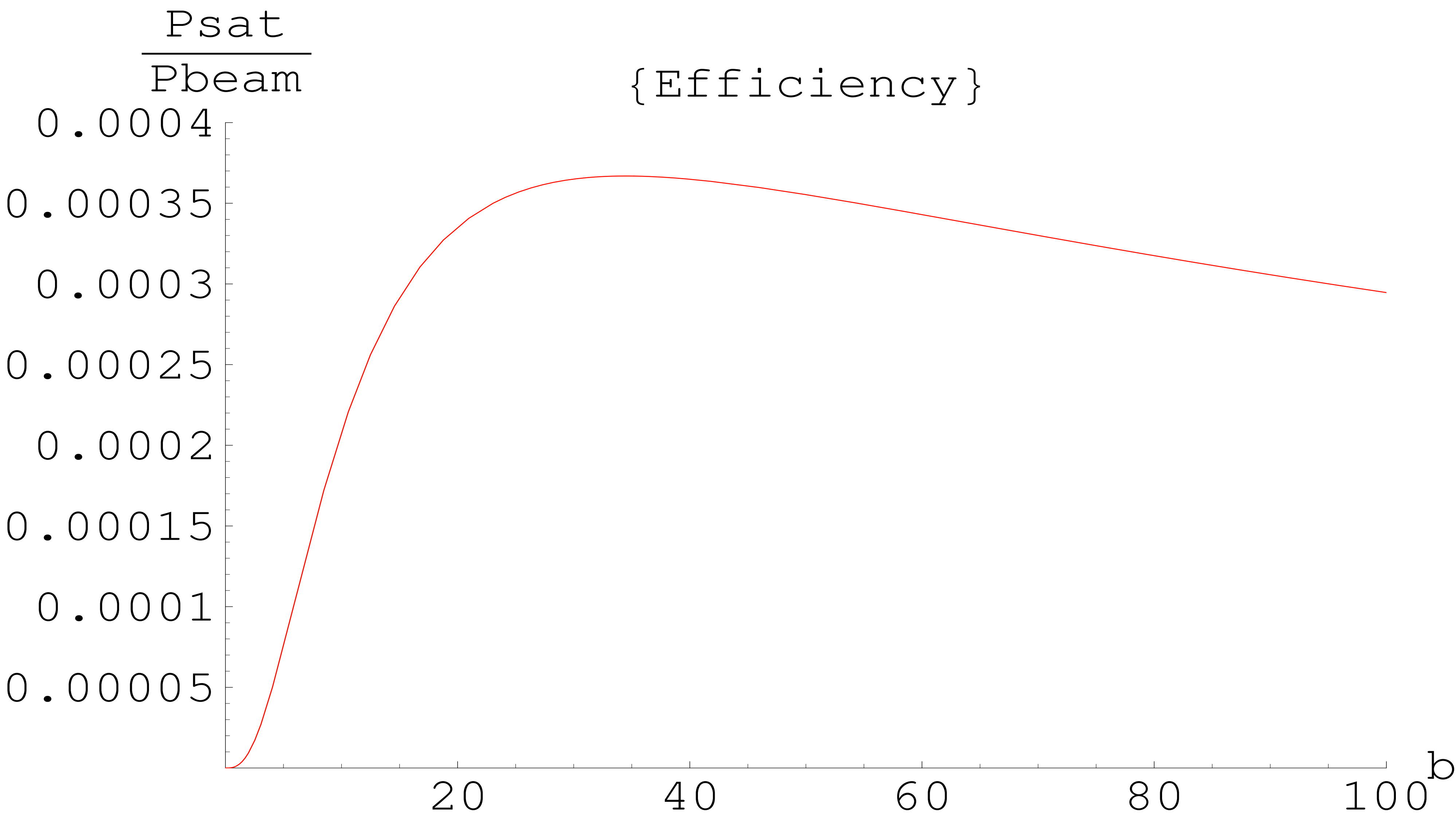
We can Plot the gainlength and the efficency as a functions of β

$$L_g = L_g(\beta) \qquad E = E(\beta)$$

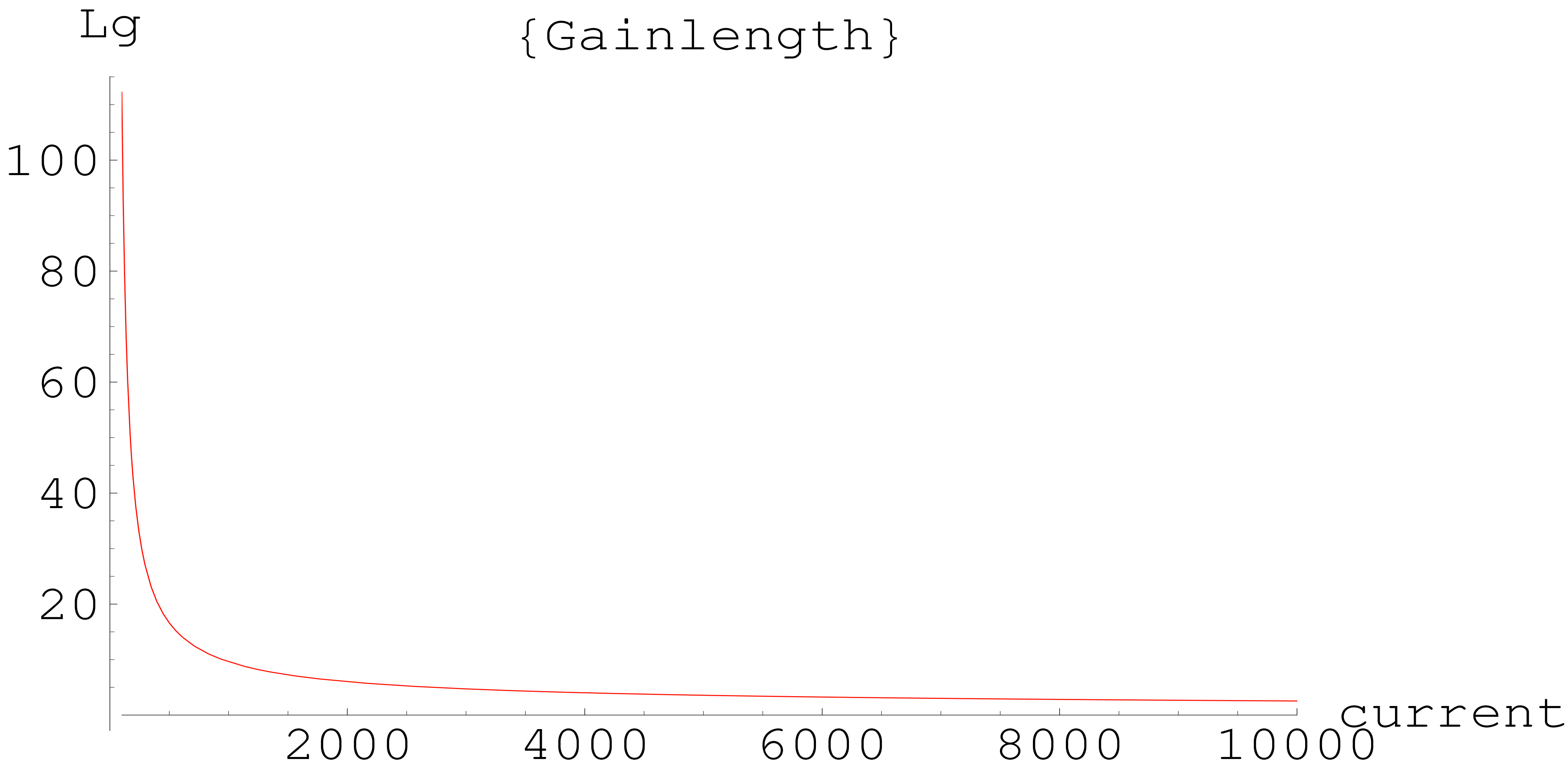
FITTING FORMULA III



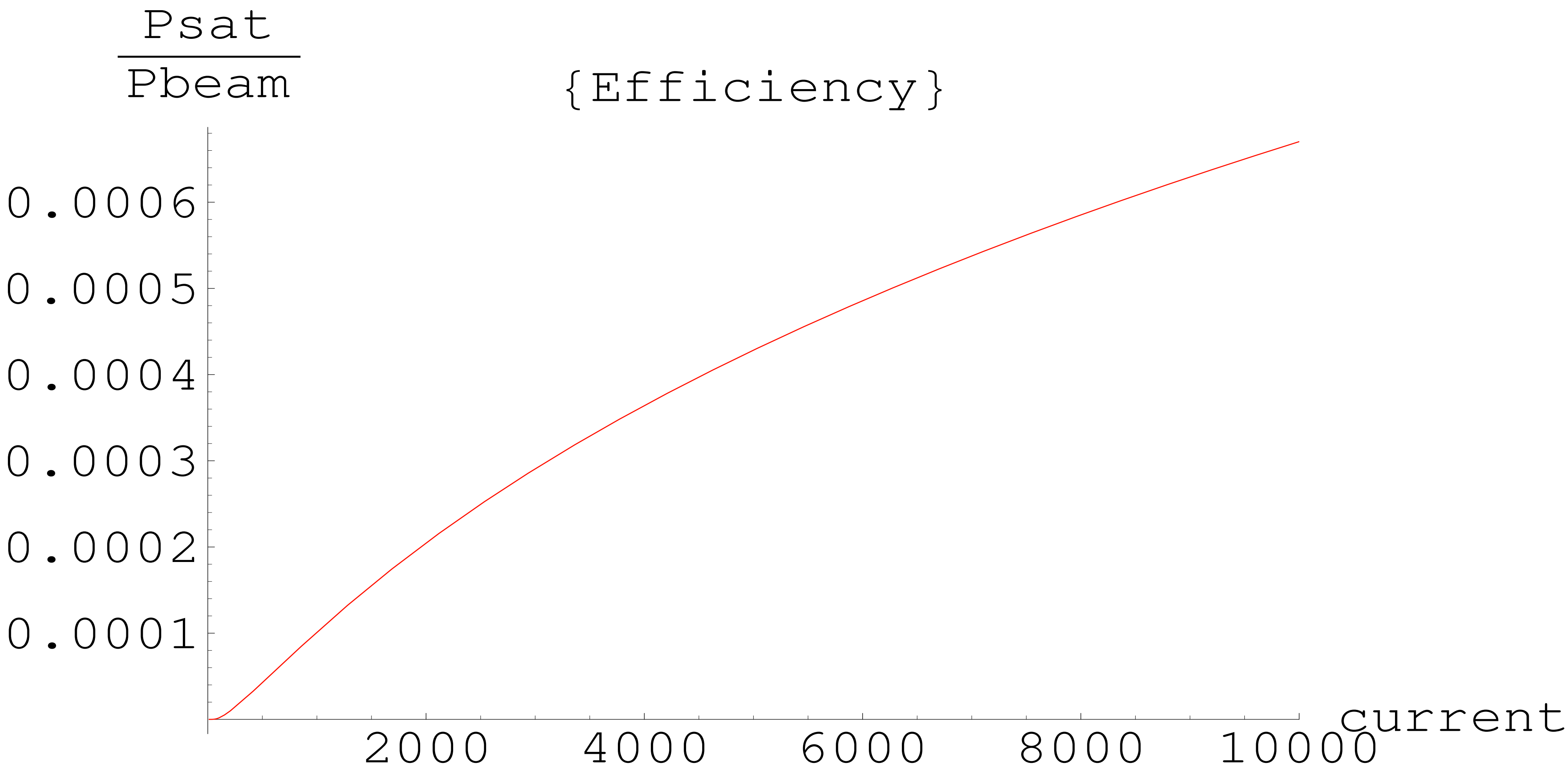
FITTING FORMULA IV



FITTING FORMULA V



FITTING FORMULA VI



Table

Table 1 High-Gain FEL experiments/projects (All SASE based except for HGHG)

Main parameters	UCLA/LANL[1]	HGHG[2]	VISA[3]	LEUTL[4]	TTF1[5]	FLASH[6]	LCLS[7]	EURO XFEL[8]
Electron energy	18 MeV	40 MeV	71 MeV	217 MeV	233 MeV	440 MeV	14.4 GeV	23 GeV
Peak current	40 ~ 170 A (100A)	110 A	250 A	200 A	400 A	2 kA	3.4 kA	5 kA
Pulse length	3-5.5 ps	4 ps	1 ps	2 ps	2.5 ps	~100 fs	200 fs	200 fs
Energy spread	0.25 %	0.043 %	0.17 %	0.2 %	0.3 MeV	0.3 MeV	0.01 %	0.01 %
Normalized emittance	2.5 ~ 4 μm (3)	4 μm	3 μm	5 μm	6 μm	4 μm	1.2 μm	1.6 μm
Undulator period	2.05 cm	3.3 cm	1.8 cm	3.3 cm	2.7 cm	2.7 cm	3 cm	4.5 cm
Undulator parameter	1.04	1.5	1.2	3.1	1.2	1.2	3.7	4
Undulator length	2 m	2 m	4 m	20 m	14 m	27 m	120 m	200 m
Average beta function	0.2 m	0.6 m	0.3 m	1.5 m	0.7 m	4.5 m	18 m	43 m
FEL wavelength	12 μm	5.3 μm	830 nm	530 nm	109 nm	32 nm	1.5 A	1 A
FEL parameter	0.016	0.01	0.008	0.004	3.2×10^{-3}	2.3×10^{-3}	5×10^{-4}	3.9×10^{-4}
Power gain length (m)	0.12	0.25	0.18	0.86	1.73	0.72	4.2	7.94
Saturation power	12 MW	27MW	76MW	50MW	24MW	17GW	17GW	32.2GW

Note: these are example numbers and see references below for details. Other notable projects not in the Table are

1. DUV-FEL (HGHG): L.-H. Yu *et al.*, Phys. Rev. Lett. **91**, 074801 (2003).
2. FERMI FEL (multi-stage HGHG): http://www.elettra.trieste.it/FERMI/index.php?n=Main.HomePage.
3. BESSY FEL (multi-stage HGHG): Technical Design Report (http://www.bessy.de/cms.php?idcatart=241).
4. Spring-8 SCSS (SASE): Conceptual Design Report (http://www-xfel.spring8.or.jp/).

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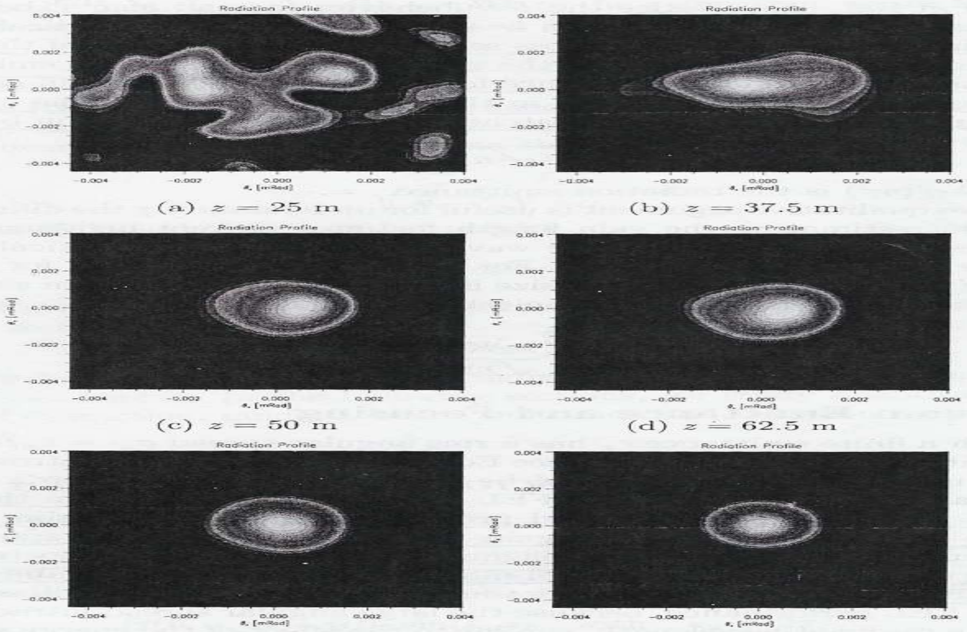
[1] M.J. Hogan *et al.*, Phys. Rev. Lett. **81**, 4867 (1998).
[2] L.-H. Yu *et al.*, Science **289**, 932 (2000).
[3] A. Tremaine *et al.*, Phys. Rev. Lett. **88**, 204801 (2002); A. Murokh *et al.*, Phys. Rev. E **67**, 066501 (2003).
[4] S. Milton *et al.*, Science **292**, 2037 (2001); Y. Li *et al.*, Phys. Rev. Lett. **89**, 234801 (2002).
[5] V. Ayvazyan *et al.*, Phys. Rev. Lett. **88**, 104802 (2002); V. Ayvazyan *et al.*, Eur. Phy. J. D **20**, 149 (2002).
[6] V. Ayvazyan *et al.*, Eur. Phy. J. D **37**, 297 (2006).
[7] Linac Coherent Light Source Conceptual Design Report, SLAC-R-593, UC-414 (2002).
[8] TESLA Techinical Design Report, DESY 2001-011, TESLA Report 2001-23, TESLA-FEL 2001-05 (2001).

TRANVERSE COHERENCE I



(a) $z = 25$ m (b) $z = 37.5$ m

(c) $z = 50$ m (d) $z = 62.5$ m

(e) $z = 75$ m (f) $z = 87.5$ m

Figure 5.2: Evolution of the LCLS radiation angular distribution at different z location (courtesy of S. Reiche, UCLA).

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TRANVERSE COHERENCE II

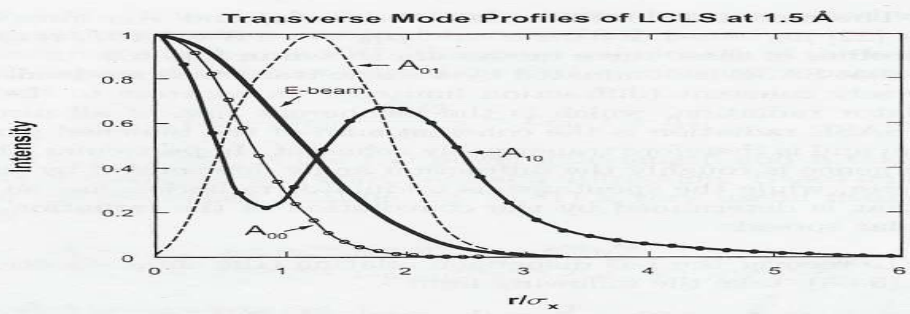

Figure 5.9: The first three dominate LCLS transverse modes (courtesy of M. Xie, LBL).

that their associated gain length is much longer. In the high-gain limit, we may keep only the fundamental mode in Eq. (5.75) and arrive at [23, 24]

$$a_\nu(R,\bar{z}) = \frac{1}{C_{00}} A_{00}(R) e^{-i\mu_{00}\bar{z}} \Big[\int d^2\bar{x}\, A_{00}(\bar{\boldsymbol{x}}) a_{0,\nu}(\bar{\nu},\bar{\boldsymbol{x}}) + \int d^2\bar{x} \int d^2\bar{p} \int d\bar{\eta} \times \bar{f}_{0,\nu}(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}},\bar{\eta}) \int_{-\infty}^{0} d\tau\, A_{00}\left(\bar{\boldsymbol{x}}^{(0)}\right) e^{i(\bar{\theta}-\mu_{00})\tau} \Big], \quad (5.104)$$

where C_{00} is the normalization given by Eq. (5.77) for the fundamental mode. The first term in the squared bracket describes the process of coherent amplification (CA), which starts from a coherent input signal $a_{0,\nu}$. The second term describes the process of self-amplified spontaneous emission (SASE), which starts from white noise. Equation (5.104) for the parallel e-beam (with vanishing emittance) reduces to those of Ref. [10, 25].

For example, using the current LCLS design parameters in Table 1, we have $\bar{\sigma}_x = 2.8$, $\bar{\sigma}_\eta = 0.45$, and $\bar{k}_\beta = 0.29$. The fundamental guided mode (A_{00} mode) has a complex growth rate $\mu_{00} = -1.2 + 0.42i$ at the optimal detuning

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