Laser Physics, Vol. 13, No. 4, 2003, pp. 611–618.

Two-Dimensional Quantum Theory of Collective Light Scattering from Bose–Einstein Condensates

N. Piovella

Dipartimento di Fisica and INFM, Universita degli Studi di Milano, Via Celoria, 16, 20133 Milano, Italy

e-mail: nicola.piovella@mi.infn.it

Received September 3, 2002

Abstract—We present a two-dimensional description of the light scattering by collective atomic recoil for a configuration in which the pump and the scattered beams are orthogonal and the atoms are in a definite state of momentum. In the quantum recoil limit and for single-photon scattering, the analytical solution can be found explicitly, both in the good-cavity limit (i.e., with no radiation losses) and in the superradiant regime (without cavity) observed in the MIT experiment.

1. INTRODUCTION

The recent observation of superradiant Rayleigh scattering from a Bose–Einstein condensate (BEC) exposed to an off-resonant laser beam [1] has motivated a series of theoretical studies aimed to describe the experiment [2–6]. In particular, in [5, 6], we have shown that the experiment described in [1] can be interpreted in terms of the collective atomic recoil laser (CARL) originally proposed by Bonifacio and coworkers [7,8]. In these original works, the atomic motion was described classically, because the system was an atomic vapor in thermal equilibrium. The recent realization of Bose–Einstein condensation in trapped alkali gases [9] offers now the possibility of realizing the CARL with a completely Doppler-free atomic system in which the atoms have a definite momentum. Moreover, because the only relevant interaction in the system is that between the atoms and the laser field, the atomic momentum changes only by discrete units of the quantum recoil momentum $\hbar\mathbf{q}$, where $\mathbf{q} = \mathbf{k}_2 - \mathbf{k}_1$ and $\mathbf{k}_{1,2}$ are the wave vectors of the scattered and incident fields. Recently, the original CARL model has been extended [10, 5, 6] to include the quantum mechanical description of the center-of-mass motion of atoms in an initial state with definite momentum, as occurs in a BEC. A similar approach was also used in [2–4] for the description of the superradiant regime. The model described in [5] assumes that the atoms back scatter the pump photons, moving parallel to the pump and scattering beams. As a consequence, the interaction is described by an 1D model. However, in the MIT experiment [1] the condensate has a cigar shape, with its major axis orthogonal to the direction of the incident beam, as shown in Fig. 1. Due to the elongated geometry of the atomic sample, two scattered beams are emitted perpendicular to the laser beam and the atoms recoil at 45° with respect to the direction of the incident laser. In this case, the geometry of the system is two-dimensional in the plane formed by the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$.

In this work, we present the full quantum two-dimensional CARL model, starting from the semiclassical model derived in [6]. Then, we solve explicitly the equations for the first scattering, both in the absence of radiation losses (as occurs in a cavity with perfectly reflecting mirrors) and in the superradiant regime (without a cavity).

2. TWO-DIMENSIONAL SEMICLASSICAL MODEL

We represent the cigar-shaped atomic sample as an ellipsoid with length L and diameter W, where $L \gg W$,

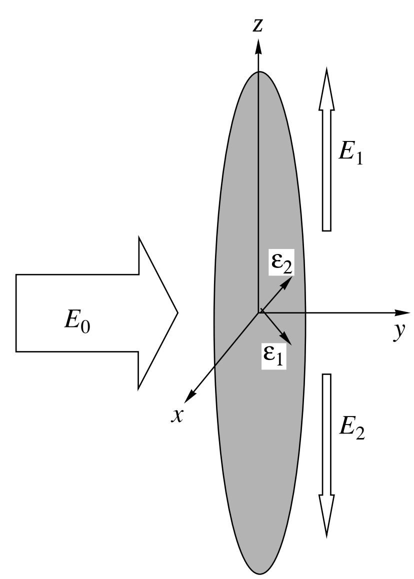

Fig. 1. Two-dimensional geometry. The condensate is exposed to a laser beam with electric field E_0 linearly polarized along the x-axis and directed along the y-axis. Two oppositely directed pulses with electric fields E_1 and E_2 are emitted along the z-axis. Also shown are the two recoil directions along the unit vectors $\hat{\varepsilon}_{1,2}$.

as shown in Fig. 1. The atomic cloud is exposed to a cw laser linearly polarized along the x-axis and directed along the y-axis, with electric field $E_i = E_0\cos[k_2(y - ct)]$. We assume that, due to the elongated shape of the atomic sample, the scattering along the positive and negative direction of the z-axis dominates over that along other directions, as has been explicitly discussed in [2], where a full multimode theory was used to show that the light-scattering mode occurs only within the geometric angle W/L. Hence, we assume that the scattered radiation consists of two oppositely directed pulses propagating along the z-axis with the same polarization of the incident field and with the electric field amplitude

$$E_s = \frac{1}{2}[E_1(z,t)e^{ik_1(z-ct)} + E_2(z,t)e^{-ik_1(z+ct)} + \text{c.c.}]. \quad (1)$$

The atomic sample is described as a collisionless gas of two-level atoms. The internal evolution of each atom is described by the density matrix elements ρ_{mn} (m, n = 1, 2) for the (1) lower and (2) upper levels. The off-diagonal element $\rho_{12} = \rho_{21}^*$ describes the x component of the dipole moment, $d_x = \mu(\rho_{1,2} + \text{c.c.})$, induced by the radiation fields, where μ is the dipole matrix element. The diagonal elements ρ_{11} and ρ_{22} describe the occupation probability of the lower and upper levels, respectively. The off-diagonal element may be described conveniently as a sum of three polarization waves:

$$\rho_{1,2} = \frac{1}{2}[S_0 e^{ik_2(y-ct)} + S_1(z,t)e^{ik_1(z-ct)} + S_2(z,t)e^{-ik_1(z+ct)} + \text{c.c.}]. \quad (2)$$

The dipole moment of each atom contributes to the macroscopic polarization of the atomic sample, $P_x = n(\mathbf{x}, t)d_x$, where $n(\mathbf{x}, t)$ is the atomic density. This polarization is a source for the radiation field via Maxwell's wave equations. Assuming that the fields $E_{1,2}$ and the polarization waves $S_{1,2}$ are slowly varying functions of z and t, then the Maxwell equations reduce to

$$\left(\frac{\partial E_1}{\partial t} + c\frac{\partial E_1}{\partial z}\right)e^{ik_1 z} + \left(\frac{\partial E_2}{\partial t} - c\frac{\partial E_2}{\partial z}\right)e^{-ik_1 z} = i\frac{ck_2\mu}{2\varepsilon_0}n(\mathbf{x},t)[S_0 e^{ik_2 y - i\Delta_{21}t} + S_1 e^{ik_1 z} + S_2 e^{-ik_1 z}], \quad (3)$$

where $\Delta_{21} = \omega_2 - \omega_1$ and $\omega_i = ck_i$. We assume that the atomic sample can be described as a collection of N point particles with positions $\mathbf{x}$, so that

$$n(\mathbf{x}, t) = \sum_{j=1}^{N}\delta^{(3)}(\mathbf{x} - \mathbf{x}_j(t)). \quad (4)$$

By multiplying both sides of Eq. (3) by $e^{\mp ik_1 z}$ and integrating over z from $z - \Delta z/2$ to $z + \Delta z/2$, where $\Delta z = \lambda_1/2$ and $\lambda_1 = 2\pi/k_1$, Eq. (3) yields

$$\left(\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial t}\right)E_{1,2} = i\frac{ck_2\mu n_a}{2\varepsilon_0}\langle S_0 e^{i(k_2 y \mp k_1 z) - i\Delta_{21}t} + S_{1,2} + S_{2,1}e^{\mp 2ik_1 z}\rangle, \quad (5)$$

where we have integrated on x and y over the section $A = \pi(W/2)^2$ of the condensate, $n_a = N/V$ is the average density, $V = A\Delta z$ is the atomic volume, and $\langle\ldots\rangle = \frac{1}{N}\sum_{j=1}^{N}(\ldots)$. In this model, the atomic center-of-mass motion is treated classically, with each atom described as a point particle with a given position and momentum. The radiation fields drive the center-of-mass motion of the atoms via the force $\mathbf{F} = d_x\nabla(E_i + E_s)$. Neglecting the fast-varying temporal terms, the equations for the center-of-mass momentum components are

$$\frac{dp_y}{dt} = \frac{ik_2\mu E_0}{4}[S_0^* + S_1^* e^{i(k_2 y - k_1 z) - i\Delta_{21}t} + S_2^* e^{i(k_2 y + k_1 z) - i\Delta_{21}t} - \text{c.c.}],$$
$$\frac{dp_z}{dt} = \frac{ik_2\mu}{4}\{S_1^* E_1 - S_2^* E_2 + S_2^* E_1 e^{2ik_1 z} - S_1^* E_2 e^{-2ik_1 z} - \text{c.c.}\} + \frac{ik_2\mu}{4}\{S_0^* e^{i\Delta_{21}t}[E_1 e^{-i(k_2 y - k_1 z)} + E_2 e^{-i(k_2 y + k_1 z)}] - \text{c.c.}\}. \quad (6)$$

We assume that the detuning $\Delta = \omega_2 - \omega_0$ between the laser field and the atomic resonance is much larger than the natural line width of the atomic transition, γ, so that the atoms remain in their lower internal energy state. Moreover, assuming that the scattering time scale is much longer than the relaxation time $1/\gamma$, we can adiabatically eliminate the atomic polarization, i.e. $S_n \approx \Omega_n/\Delta$, where $\Omega_n = \mu E_n/\hbar$ is the Rabi frequency for the n-th field and n = 0, 1, 2. With these approximations, Eqs. (6) become

$$\frac{dp_y}{dt} = i\hbar k_2\frac{\Omega_0}{4\Delta}[\tilde{\Omega}_1^* e^{i(k_2 y - k_1 z)} + \tilde{\Omega}_2^* e^{i(k_2 y + k_1 z)} - \text{c.c.}],$$
$$\frac{dp_z}{dt} = i\hbar k_2\frac{\Omega_0}{4\Delta}[\tilde{\Omega}_1 e^{i(k_1 z - k_2 y)} + \tilde{\Omega}_2^* e^{i(k_1 z + k_2 y)} - \text{c.c.}] + \frac{i\hbar k_2}{2\Delta}[\tilde{\Omega}_1\tilde{\Omega}_2^* e^{2ik_1 z} - \text{c.c.}], \quad (7)$$

where $\tilde{\Omega}_{1,2} = \Omega_{1,2}e^{i\Delta_{21}t}$. We observe that the interference between the pump and the two scattered fields forms two two-dimensional pendulum potentials, $V_{1,2}(y, z) \propto E_0 E_{1,2}\cos(\mathbf{q}_{1,2}\cdot\mathbf{x} + \phi_{1,2})$ in the plane (y, z), where $\mathbf{q}_{1,2} = k_2\hat{y} - k_1\hat{z} \cong q\hat{\varepsilon}_{1,2}$, $q = \sqrt{2}k_2$, $\hat{\varepsilon}_{1,2} = (\hat{y} \mp \hat{z})/\sqrt{2}$ are unit vectors and $\phi_{1,2}$ are the phases of the complex amplitudes $E_{1,2}$. A third one-dimensional pendulum potential $V_3(z) \propto E_1E_2\cos(2k_1z + \phi_1 - \phi_2)$ forms along the z-axis due to the interference between the two counter-propagating scattered fields. Eqs. (5) and (7) may conveniently be written in the following dimensionless form [6]:

$$\frac{d\vartheta_{1,2}}{d\tau} = \frac{2}{\rho}p_{1,2},$$
$$\frac{dp_{1,2}}{d\tau} = g[a_{1,2}e^{-i\vartheta_{1,2}} + \text{c.c.}] \mp \frac{ig}{a_0}[a_{1,2}a^*_{2,1}e^{i(\vartheta_2-\vartheta_1)} - \text{c.c.}], \quad (8)$$
$$\frac{\partial a_{1,2}}{\partial\tau} \pm \frac{\partial a_{1,2}}{\partial\xi} = gN\left\langle e^{i\vartheta_{1,2}} + \frac{ia_{2,1}}{a_0}e^{i(\vartheta_2-\vartheta_1)}\right\rangle + i\left(\delta + \frac{1}{a_0}\right)a_{1,2},$$

where $g = \sqrt{\rho/2N}$, $\vartheta_{1,2} = k_2y \mp k_1z \approx \mathbf{q}_{1,2}\cdot\mathbf{x}$, $p_{1,2} = (p_y \mp p_z)/\hbar k_2 \approx 2\mathbf{p}\cdot\hat{\varepsilon}_{1,2}/\hbar q$, $a_{1,2} = -i\sqrt{\varepsilon_0 V/2\hbar\omega_2}E_{1,2}e^{i\delta\tau}$, $\tau = \rho\omega_r t$, $\xi = \rho\omega_r z/c$, $\delta = \Delta_{21}/\rho\omega_r$, $\rho\omega_r$ is the collective recoil bandwidth, $\omega_r = \hbar q^2/2m_a$ is the recoil frequency, and $\rho = (\Omega_0/2\Delta_0)^{2/3}(\omega_2\mu^2 n_a/\hbar\varepsilon_0\omega_r^2)^{1/3}$ is the collective CARL parameter [7].

In the mean-field limit, we can approximate the spatial derivative in the field equation by a linear loss term:

$$\frac{da_{1,2}}{d\tau} + \kappa a_{1,2} = gN\left\langle e^{i\vartheta_{1,2}} + \frac{ia_{2,1}}{a_0}e^{i(\vartheta_2-\vartheta_1)}\right\rangle + i\left(\delta + \frac{1}{a_0}\right)a_{1,2}, \quad (9)$$

where $\kappa = c/2L\rho\omega_r$ and L/c is the transit time of the photon along the major axis of the condensate. With this approximation, the finite interaction time due to the escape of radiation from the atomic sample is described by an incoherent decay of the field amplitude in the sample at a rate $c/2L$, which is half the inverse of the radiation 'lifetime' in the atomic sample. If the radiation losses can be neglected (as for instance in a good optical cavity), then Eqs. (8) and (9), with $\kappa = 0$, may be derived from the following Hamiltonian:

$$H = \sum_{j=1}^{H}\left\{\frac{1}{\rho}(p_{1j}^2 + p_{2j}^2) + ig(a_1^*e^{i\vartheta_{1j}} + a_2^*e^{i\vartheta_{21j}} - \text{c.c.}) - \frac{g}{a_0}[a_1a_2^*e^{i(\vartheta_{21j}-\vartheta_{1j})} + \text{c.c.}]\right\} - \left(\delta + \frac{1}{a_0}\right)(a_1^*a_1 + a_2^*a_2) = \sum_{j=1}^{N}H_j(\vartheta_{1j}, \vartheta_{2j}, p_{1j}, p_{2j}). \quad (10)$$

3. TWO-DIMENSIONAL QUANTUM MODEL

In quantum theory, the variables θ_1, θ_2, p_1, p_2, a_1, and a_2 in the Hamiltonian (10) must be treated as quantum operators satisfying the canonical commutation rules $[\vartheta_{\alpha j}, p_{\beta k}] = i\delta_{\alpha\beta}\delta_{jk}$ and $[a_\alpha, a_\beta^+] = \delta_{\alpha\beta}$ wit, $\alpha, \beta = 1, 2$ and $j, k = 1, \dots, N$. The single-particle Hamiltonian $H(\theta_1, \theta_2, p_1, p_2)$ in (10) can be second-quantized as

$$\hat{H} = \int_0^{2\pi}d\vartheta_1\int_0^{2\pi}d\vartheta_2\hat{\Psi}^+(\vartheta_1, \vartheta_2) \times H\left(\vartheta_1, \vartheta_2, -i\frac{\partial}{\partial\vartheta_1}, -i\frac{\partial}{\partial\vartheta_2}\right)\hat{\Psi}(\vartheta_1, \vartheta_2), \quad (11)$$

where the atomic-field operators obey the usual bosonic equal time commutation relations $[\hat{\Psi}(\vartheta_1, \vartheta_2), \hat{\Psi}^+(\vartheta_1', \vartheta_2')] = \delta(\vartheta_1 - \vartheta_1')\delta(\vartheta_2 - \vartheta_2')$, $[\hat{\Psi}(\vartheta_1, \vartheta_2), \hat{\Psi}(\vartheta_1', \vartheta_2')] = 0$ and the normalization condition $\int_0^{2\pi}d\vartheta_1\int_0^{2\pi}d\vartheta_2\hat{\Psi}^+(\vartheta_1, \vartheta_2)\hat{\Psi}(\vartheta_1, \vartheta_2) = \hat{N}$. We introduce creation and annihilation operators for the two components of the atomic momentum p_1 and p_2, i.e., $\hat{\Psi}(\vartheta_1, \vartheta_2) = \sum_{mn}c(m, n)\langle\vartheta_1|m\rangle_1\langle\vartheta_2|n\rangle_2$, where $p_\alpha|m\rangle_\alpha = m|m\rangle_\alpha$ (with $m = -\infty, \dots, \infty$ and $\alpha = 1, 2$), $\langle\vartheta_\alpha|m\rangle_\alpha = \frac{1}{\sqrt{2\pi}}\exp(im\vartheta_\alpha)$ and $c(m, n)$ are bosonic operators obeying the commutation rules $[c(m, n), c^+(m', n')] = \delta_{mm'}\delta_{nn'}$ and $[c(m, n), c(m', n')] = 0$. We assume that the atoms are delocalized inside the condensate and that, at zero temperature, the momentum uncertainty $\sigma_p \approx \hbar/L$ is negligible compared to the recoil momentum $\hbar q$; i.e., we assume that $L \gg \lambda_2$. Then, from the field equations $i\partial_\tau\hat{\Psi} = [\hat{H}, \hat{\Psi}]$ and $i\partial_\tau a = [\hat{H}, a]$, it is easy to derive the equations for $c(m, n)$, a_1, and a_2. For simplicity, we assume that $a_0 \gg a_{1,2}$, so that the terms proportional to $1/a_0$ in (10) can be neglected. We obtain

$$\frac{dc(m,n)}{d\tau} = -i\left\{\frac{1}{\rho}(m^2 + n^2) - \frac{\delta}{N}(a_1^+a_1 + a_2^+a_2)\right\}c(m,n) + g\{a_1^+c(m-1,n) + a_2^+c(m,n-1) - a_1c(m+1,n) - a_2c(m,n+1)\}, \quad (12)$$

$$\frac{da_1}{d\tau} = g\sum_{m,n} c^+(m,n)c(m-1,n) + i\delta a_1 - \kappa a_1, \quad (13)$$

$$\frac{da_2}{d\tau} = g\sum_{m,n} c^+(m,n)c(m,n-1) + i\delta a_2 - \kappa a_2. \quad (14)$$

In the following, we will solve Eqs. (12)—(14) while considering the operators $c(m, n)$, a_1, and a_2 as classical variables. Furthermore, we have added phenomenological decay terms $-\kappa a_{1,2}$ to the field equations to take into account field losses due to propagation out of the atomic medium.

4. LINEAR ANALYSIS

Let consider the equilibrium state with no scattered fields, $a_1 = a_2 = 0$, and all the atoms in the same momentum state n_0, i.e., $c(m, n) = \sqrt{N}e^{-in_0^2\tau/\rho}\delta_{mm_0}\delta_{nn_0}$. This is equivalent to assuming that the temperature is zero and all the atoms move with the same momentum $n_0(\hbar q)$ along the y-axis. It has been demonstrated [5, 10] that the system is unstable for certain value of the detuning δ. In fact, by linearizing around this equilibrium state, Eqs. (12)—(14) decouple into two sets of three equations:

$$\begin{aligned}
\frac{dc_+^{(\alpha)}}{d\tau} &= -i\delta_+ c_+^{(\alpha)} + \sqrt{\frac{\rho}{2}}\tilde{a}_\alpha, \\
\frac{dc_-^{(\alpha)}}{d\tau} &= -i\delta_- c_-^{(\alpha)} - \sqrt{\frac{\rho}{2}}\tilde{a}_\alpha, \\
\frac{d\tilde{a}_\alpha}{d\tau} &= \sqrt{\frac{\rho}{2}}(c_-^{(\alpha)} + c_+^{(\alpha)}) - \kappa a_\alpha,
\end{aligned} \quad (15)$$

where $\tilde{a}_\alpha = a_\alpha e^{-i\delta\tau}$, $\alpha = 1, 2$, $c_-^{(1)} = c(n_0 - 1, n_0)e^{2in_0^2\tau/\rho - i\delta\tau}$, $c_+^{(1)} = c^*(n_0 + 1, n_0)e^{-2in_0^2\tau/\rho - i\delta\tau}$, $c_-^{(2)} = c(n_0, n_0 - 1)e^{2in_0^2\tau/\rho - i\delta\tau}$, $c_+^{(2)} = c^*(n_0, n_0 + 1)e^{-2in_0^2\tau/\rho - i\delta\tau}$, and $\delta_\pm = \delta - \frac{2n_0}{\rho} \mp \frac{1}{\rho}$. We note that the atoms have the same probability of scattering a pump photon along the positive or negative direction of the z-axis, respectively, into the mode a_1 or a_2 and of recoiling along the directions of $\hat{\varepsilon}_1$ or $\hat{\varepsilon}_2$. Although the atom can in principle absorb a photon either from the pump or from the probe, in [5] it has been shown that, for $\rho < 1$ when $\kappa = 0$ (good cavity regime) or for $\rho < \sqrt{2\kappa}$ when $\kappa > 1$ (superradiant regime), the atom can only absorb a photon from the pump and emit it into the probe. This is because, in these limits, the gain bandwidth is smaller than the recoil energy shift $\hbar\omega_r$ experienced by the atom when it scattered a single photon.

5. NONLINEAR REGIME

It is useful to introduce the two-dimensional density matrix

$$S(m,n,m',n') = \frac{1}{N}c^*(m,n)c(m',n')e^{i(m+n-m'-n')\delta\tau} \quad (16)$$

and the fields $A_{1,2} = (2/\rho N)^{1/2}a_{1,2}e^{-i\delta\tau}$. A straightforward calculation yields, from (12)–(14), the following set of equations:

$$\begin{aligned}
\frac{dS(m,n,m',n')}{d\tau} &= -i[(m-m')\delta_{m,m'} + (n-n')\delta_{n,n'}]S(m,n,m',n') \\
&+ \frac{\rho}{2}\{A_1[S(m-1,n,m',n') - S(m,n,m'+1,n')] \\
&+ A_1^*[S(m,n,m'-1,n') - S(m+1,n,m',n')] \\
&+ A_2[S(m,n-1,m',n') - S(m,n,m',n'+1)] \\
&+ A_2^*[S(m,n,m',n'-1) - S(m,n+1,m',n')]\},
\end{aligned} \quad (17)$$

$$\begin{aligned}
\frac{dA_1}{d\tau} &= \sum_{m,n} S(m,n,m-1,n) - \kappa A_1, \\
\frac{dA_2}{d\tau} &= \sum_{m,n} S(m,n,m,n-1) - \kappa A_2,
\end{aligned} \quad (18)$$

where $\delta_{m,m'} = \delta - (m + m')/\rho$. Although Eqs. (17) and (18) can be solved numerically, it is worth studying the single-scattering regime analytically, in which the atoms scatter only one photon into the modes a_1 and a_2, making transitions from the initial level (0, 0) to the three levels (1, 0), (0, 1), and (1, 1), as shown schematically in Fig. 2 (we set $n_0 = 0$ in order to simplify the notations). The occupation probabilities of the momentum level (α, β) are defined as $p(\alpha, \beta) = S(\alpha, \beta, \alpha, \beta)$ and, from (17), satisfy the equations

$$\frac{dp(0,0)}{d\tau} = -\frac{\rho}{2}[A_1S(0,0,1,0) + A_2S(0,0,0,1)] + \text{c.c.},$$

$$\frac{dp(1,0)}{d\tau} = \frac{\rho}{2}[A_1S(0,0,1,0) - A_2S(1,0,1,1)] + \text{c.c.},$$

$$\frac{dp(0,1)}{d\tau} = -\frac{\rho}{2}[A_1S(0,1,1,1) - A_2S(0,0,0,1)] + \text{c.c.},$$

$$\frac{dp(1,1)}{d\tau} = \frac{\rho}{2}[A_1S(0,1,1,1) + A_2S(1,0,1,1)] + \text{c.c.}$$

We note that $p(0, 0) + p(0, 1) + p(1, 0) + p(1, 1) = 1$. We write also the equations for the six transition elements $S(0, 0, 1, 0)$, $S(0, 0, 0, 1)$, $S(1, 0, 1, 1)$, $S(1, 0, 1, 1)$, $S(0, 0, 1, 1)$, and $S(0, 1, 1, 0)$:

$$\frac{dS(0,0,1,0)}{d\tau} = i\delta_+ S(0,0,1,0) + \frac{\rho A_1^*}{2}[p(0,0) - p(1,0)] - \frac{\rho}{2}[A_2 S(0,0,1,1) + A_2^* S(0,1,1,0)],$$

$$\frac{dS(0,0,0,1)}{d\tau} = i\delta_+ S(0,0,0,1) + \frac{\rho A_2^*}{2}[p(0,0) - p(0,1)] - \frac{\rho}{2}[A_1 S(0,0,1,1) + A_1^* S(1,0,0,1)],$$

$$\frac{dS(1,0,1,1)}{d\tau} = i\delta_+ S(1,0,1,1) + \frac{\rho A_2^*}{2}[p(1,0) - p(1,1)] + \frac{\rho}{2}[A_1 S(0,0,1,1) + A_1^* S(1,0,0,1)],$$

$$\frac{dS(0,1,1,1)}{d\tau} = i\delta_+ S(0,1,1,1) + \frac{\rho A_1^*}{2}[p(0,1) - p(1,1)] + \frac{\rho}{2}[A_2 S(0,0,1,1) + A_2^* S(0,1,1,0)],$$

$$\frac{dS(0,0,1,1)}{d\tau} = 2i\delta_+ S(0,0,1,1) + \frac{\rho}{2}[A_1^* S(0,0,0,1) - A_1^* S(1,0,1,1) + A_2^* S(0,0,1,0) - A_2^* S(0,1,1,1)],$$

$$\frac{dS(0,1,1,0)}{d\tau} = \frac{\rho}{2}[A_1^* S(0,1,0,0) - A_1^* S(1,1,1,0) + A_2 S(0,0,1,0) - A_2 S(0,1,1,1)].$$

These equations, together with the equations for the two radiation modes,

$$\frac{dA_1}{d\tau} = S(1,0,0,0) + S(1,1,0,1) - \kappa A_1,$$

$$\frac{dA_2}{d\tau} = S(0,1,0,0) + S(1,1,1,0) - \kappa A_2,$$

form a closed system. Assuming $\delta_+ = 0$ (i.e. $\omega_2 = \omega_1 + \omega_r$) and symmetrical initial conditions for the modes 1 and 2, then $A_1 = A_2 = A$ and $p(0, 1) = p(1, 0)$. These conditions imply that $S(0, 0, 1, 0) = S(0, 0, 0, 1) = S_1$, $S(1, 0, 1, 1) = S(0, 1, 1, 1) = S_2$, and $S(0, 1, 1, 0) = S(0, 0, 1, 1) = S_3$ are real. Defining $p_0 = p(0, 0)$, $p_1 = p(0, 1)$, and $p_2 = p(1, 1)$, the system reduces to the following form:

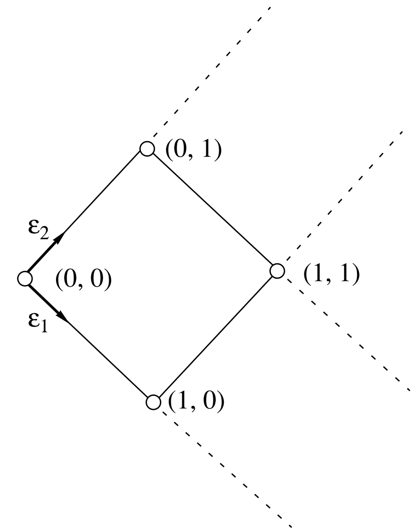

Fig. 2. The momentum lattice for the states (m, n). The first scattering includes transition from (0, 0) to (0, 1), (0, 1), and (1, 1).

$$\begin{aligned}
\frac{dS_1}{d\tau} &= \frac{\rho}{2}A(p_0 - p_1) - \rho A S_3, \\
\frac{dS_2}{d\tau} &= \frac{\rho}{2}A(p_1 - p_2) + \rho A S_3, \\
\frac{dS_3}{d\tau} &= \rho A(S_1 - S_2), \\
\frac{dp_0}{d\tau} &= -2\rho A S_1, \\
\frac{dp_1}{d\tau} &= \rho A(S_1 - S_2), \\
\frac{dp_2}{d\tau} &= 2\rho A S_2, \\
\frac{dA}{d\tau} &= S_1 + S_2 - \kappa A.
\end{aligned} \tag{19}$$

By introducing $S_\pm = S_1 \pm S_2$, $W_+ = p_0 - p_2$, and $W_- = p_0 - 2p_1 + p_2 = 1 - 4p_1$ (where we used $p_0 + 2p_1 + p_2 = 1$), Eqs. (19) can be divided into two systems:

$$\begin{aligned}
\frac{dS_+}{d\tau} &= \frac{\rho}{2}AW_+, \\
\frac{dW_+}{d\tau} &= -2\rho A S_+, \\
\frac{dA}{d\tau} &= S_+ - \kappa A,
\end{aligned} \tag{20}$$

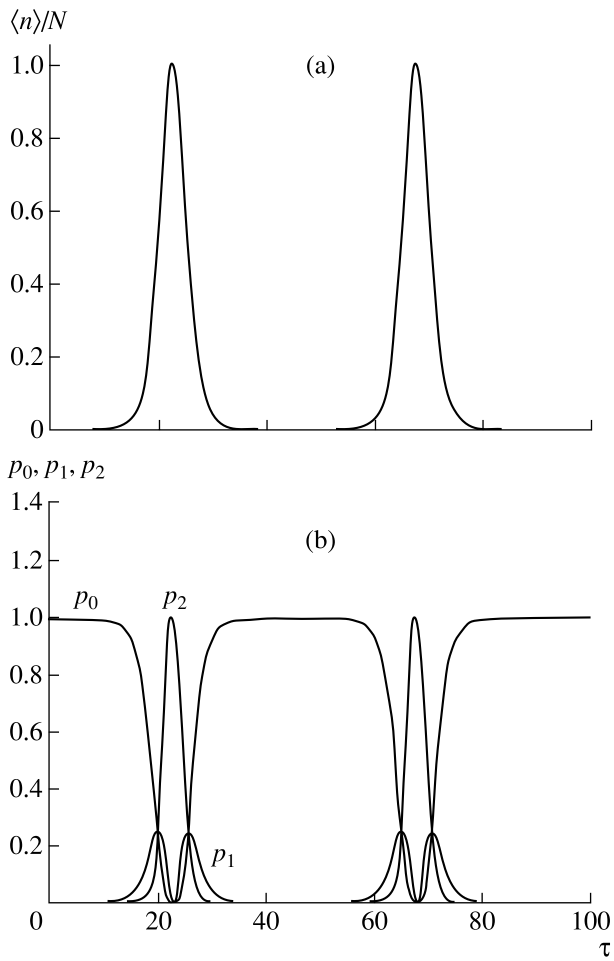

Fig. 3. (a) Average number of scattered photons per atom, $\langle n\rangle/N$, (b) and population probabilities p_0, p_1, and p_2 as a function of τ, obtained by integrating Eqs. (17) and (18) numerically for $\rho = 0.2$, $\delta = 5$, and $\kappa = 0$.

$$\frac{dS_-}{d\tau} = \frac{\rho}{2}AW_- - 2\rho A S_3,$$
$$\frac{dW_-}{d\tau} = -4\rho A S_-, \qquad (21)$$
$$\frac{dS_3}{d\tau} = \rho A S_-.$$

We note that the variables A, S_+, and W_+ form a closed system, formally equivalent to the Maxwell–Bloch equations for a two-level system interacting with a single-mode radiation field [11]. In the present case, the transition occurs between the levels (0, 0) and (1, 1), passing through the intermediate levels (0, 1) and (1, 0). The occupation probability for the level (0, 1) is completely determined by the field A, as shown using Eqs. (21), which can be solved by observing that they admit two constants of motion, $W_-^2 + 8(S_-^2 + 2S_3^2) = 1$ and $W_- + 4S_3 = 1$. Eliminating S_3 and defining $W' = 2W_- - 1 = 1 - 8p_1$, we obtain $16S_-^2 + W'^2 = 1$. Finally, defining $W' = \cos\phi_-$ and $4S_- = \sin\phi_-$, we obtain $d\phi_-/d\tau = 2\rho A$. The field A can be obtained from Eqs. (20): we define the Bloch angle ϕ_+ such that $W_+ = \cos\phi_+$ and $2S_+ = \sin\phi_+$, with $\phi_+(0) = 0$. Then, the first two equations of (20) give $d\phi_+/d\tau = \rho A$, so that $\phi_- = 2\phi_+$. The third equation of (20) gives

$$\frac{d^2\phi_+}{d\tau^2} + \kappa\frac{d\phi_+}{d\tau} - \frac{\rho}{2}\sin\phi_+ = 0. \qquad (22)$$

Expressing p_0, p_1, and p_2 as a function of the Bloch angle ϕ_+, we obtain

$$p_0 = \frac{1}{4}(1+\cos\phi_+)^2,$$
$$p_1 = \frac{1}{4}\sin^2\phi_+, \qquad (23)$$
$$p_2 = \frac{1}{4}(1-\cos\phi_+)^2.$$

We observe that, when $p_0 = p_2 = 1$ for $\phi_+ = 0$ and $\phi_+ = \pi$, respectively, $p_1 = 0$, whereas p_1 is maximum for $\phi_+ = \pi/2$, with $p_0 = p_1 = p_2 = 1/4$. We consider now the two cases of interest:

(i) **good cavity limit;** $\kappa = 0$.

The solution to Eq. (22) is [11] $\cos\phi_+ = 2\tanh^2[\sqrt{\rho/2}\,(\tau-\tau_D)] - 1$, where $\tau_D = \sqrt{2/\rho}\,\ln(\rho/2)$ is the delay time. Substituting it into Eqs. (23), we obtain

$$p_0 = \tanh^4[\sqrt{\rho/2}(\tau-\tau_D)],$$
$$p_1 = \tanh^2[\sqrt{\rho/2}(\tau-\tau_D)]\,\mathrm{sech}^2[\sqrt{\rho/2}(\tau-\tau_D)], \qquad (24)$$
$$p_2 = \mathrm{sech}^4[\sqrt{\rho/2}(\tau-\tau_D)],$$

whereas the average number of scattered photons is $\langle n\rangle = N\,\mathrm{sech}^2(\sqrt{\rho/2(\tau-\tau_D)})$. In Fig. 3, we report the results of the numerical integration of Eqs. (17) and (18) for $\rho = 0.2$, $\delta = 5$, and $\kappa = 0$, showing (a) the average number of photons scattered per atom, $\langle n\rangle/N$, and (b) the population probabilities p_0, p_1, and p_2 as a function of τ. The maximum number of scattered photons is equal to the number of atoms N. In this case, the atoms oscillate between the levels (0, 0) and (1, 1).

(ii) **superradiant limit;** $\kappa \gg \sqrt{2\rho}$.

The solution is that for an overdamped pendulum starting from an unstable equilibrium, with $\cos\phi_+ = \tanh[\sqrt{\rho/2\kappa}\,(\tau-\tau_D)]$. Substituting it into Eqs. (23), we obtain

$$p_0 = \frac{1}{4}\{1+\tanh[\sqrt{\rho/2\kappa}(\tau-\tau_D)]\}^2,$$
$$p_1 = \frac{1}{4}\mathrm{sech}^2[\sqrt{\rho/2\kappa}(\tau-\tau_D)], \qquad (25)$$

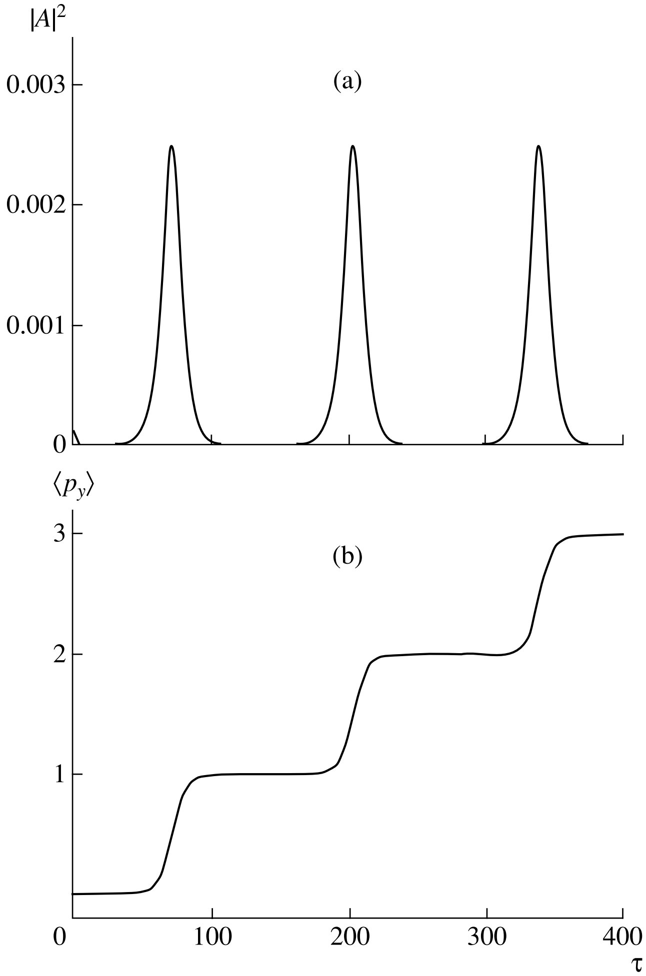

Fig. 4. Sequential superradiant regime: (a) scaled intensity $|A|^2$ and (b) the average momentum $\langle p_y \rangle$ in units of $\hbar k_2$ as a function of τ, obtained by integrating Eqs. (17) and (18) for $\rho = 2$, $\delta = 0.5$, $\kappa = 10$, and $A(0) = 0.01$.

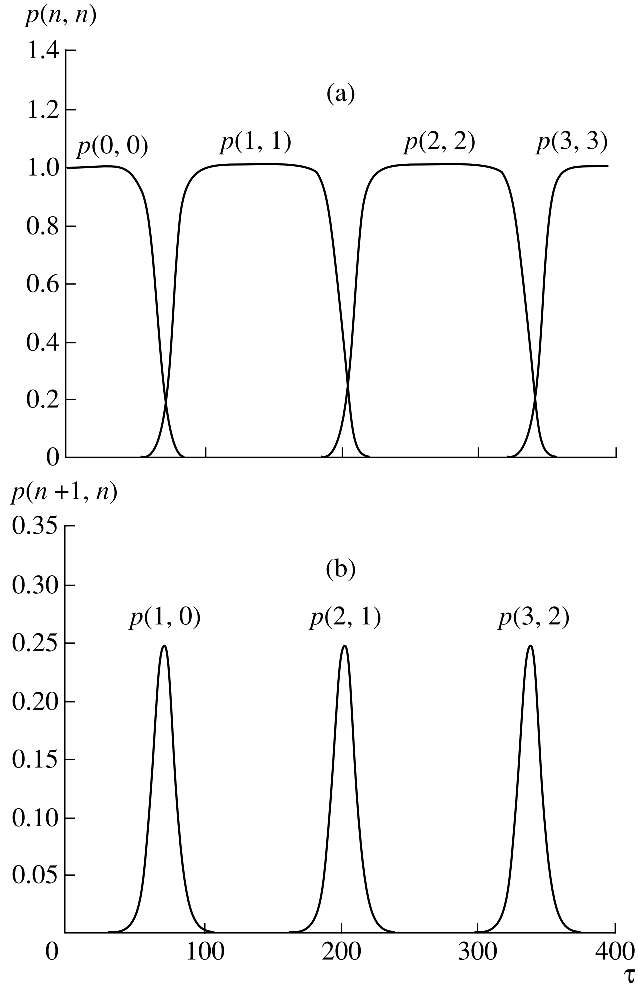

Fig. 5. Sequential superradiant regime: population probabilities (a) $p(n, n)$ and (b) $p(n + 1, n)$ for $n = 0, 1$, and 2 as a function of τ for the same parameters as in Fig. 4.

$$p_2 = \frac{1}{4}\{1 - \tanh[\sqrt{\rho/2\kappa}(\tau - \tau_D)]\}^2.$$

The average number of scattered photons is $\langle n \rangle = \frac{3N^2\gamma\tau_c}{8\pi}\left(\frac{\lambda^2}{A}\right)\left(\frac{\Omega_0}{2\Delta_0}\right)^2 \mathrm{sech}^2[\sqrt{\rho/2\kappa}(\tau - \tau_D)]$, where $\tau_c = L/c$ is the photon lifetime and A is the condensate section. Figure 4 shows the results of the numerical integration of Eqs. (17) and (18) for $\rho = 2$, $\delta = 0.5$, $\kappa = 10$, and $A(0) = 0.01$. Figures 4a and 4b show the scaled intensity $|A|^2$ and the average y component of the momentum $\langle p_y \rangle$ in units of $\hbar k_2$, whereas Figs. 5a and 5b show the population $p(n, n)$ and $p(n + 1, n)$ for $n = 0, 1, 2$, and 3. We observe a sequential superradiant scattering, in which the atoms, initially in the state (0, 0), change their y component of the momentum in discrete steps of $\hbar k_2$, emitting a superradiant pulse during each quantum jump.

6. DISCUSSION

Our results show that, in the superradiant regime and when the atomic decoherence is neglected, it is possible to transfer completely the population of the initial motional level (0, 0) to the level (1, 1). The process can be continued with the transfer from (1, 1) to (2, 2), and so on. In this way, the atoms scatter the average number of photons along the two opposite directions of the long axis of the condensate, populating the levels (n_0, n_0) (with $n_0 = 0, 1, \ldots$) sequentially and the adjacent levels $(n_0 + 1, n_0)$ and $(n_0, n_0 + 1)$ only transitorily with a maximum probability of 1/4. In a real system, as in [1], decoherence is always present; this can make the transition between the motional levels less efficient. As a consequence, all the sites (m, n) of the momentum lattice become populated, depending on the ratio between the gain and the decoherence rate. In a future work, we will generalize the analytical solution obtained here to include decoherence effects. We comment here shortly on the work performed in [3], where the authors derived a model for the four level (0, 0), (0, 1), (1, 0), and (1, 1) similar to our model. In [3], the authors introduced a factorization of the average product of operators different from the usual factorization $\langle AS_i \rangle = \langle A \rangle \langle S_i \rangle$ (where $i = 1, 2, 3$) adopted here. Then, by neglecting higher order terms, they arrived ato a closed set of equations for a three-level system. It should be noted that the reduction to only four motional states is valid only for a small coupling constant (in the superradiant case, for $\omega_r \rho^3 < c/L$). The oscillating behavior for large coupling rates observed in [3] is probably due to the violation of this condition. Moreover, the numerical integration of the exact Eqs. (17) and (18) shows that, in this limit, the levels (2, 0) and (0, 2) remain empty, so that, in the four-level approximation, the decay term introduced in [3] in the equation for $p(1, 0)$ in order to compensate for the transition toward the excluded level (2, 0) is not necessary, nor can its rate be assumed to be equal to the field decay rate κ, which is instead proportional to c/L.

REFERENCES

1. Inouye, S., Chikkatur, A.M., Stamper-Kurn, D.M., *et al.*, 1999, *Science*, **285**, 571.
2. Moore, G.M. and Meystre, P., 1999, *Phys. Rev. Lett.*, **83**, 5202.
3. Mustecaplioglu, O.E. and You, L., 2000, *Phys. Rev. A*, **62**, 063615.
4. Trifonov, E.D., 2001, *JETP*, **93**, 969.
5. Piovella, N., Gatelli, M., and Bonifacio, R., 2001, *Opt. Commun.*, **194**, 167.
6. Piovella, N., Gatelli, M., Martinucci, L., *et al.*, 2002, *Laser Phys.*, **12**, 1.
7. Bonifacio, R., De Salvo Souza, L., Narducci, L.M., and D'Angelo, E.J., 1994, *Phys. Rev. A*, **50**, 1716.
8. Bonifacio, R. and De Salvo Souza, L., 1995, *Appl. Phys. B*, **60**, S233.
9. Courteille, Ph., Bagnato, V.S., and Yukalov, V.I., 2001, *Laser Phys.*, **11**, 659.
10. Moore, G.M. and Meystre, P., 1998, *Phys. Rev. A*, **58**, 3248.
11. Bonifacio, R. and Preparata, G., 1970, *Phys. Rev. A*, **2**, 336.