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Entanglement in a Bose–Einstein condensate by collective atomic recoil

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Abstract

We address the interaction between a Bose–Einstein condensate and a single-mode quantized radiation field in the presence of a strong far off-resonant pump laser. The generation of atom–atom and atom–field entanglement is demonstrated in the linear regime. The effects of cavity losses are taken into account and an analytic solution of the corresponding master equation is given in terms of the Wigner function of the system.

1. Introduction

The experimental realization of Bose–Einstein condensation opened the possibility of generating macroscopic atomic fields whose quantum statistical properties can in principle be manipulated and controlled, very much like those of quantum-optical fields [1]. The system usually considered for this purpose is a Bose–Einstein condensate (BEC) driven by a far off-resonant pump laser of frequency ω and wave vector $\vec{k}$ and coupled to a single mode in an optical ring cavity. This results in an exponential gain of the cavity mode and in the spontaneous formation of a density grating (bunching) in the BEC, as described in the collective atomic recoil lasing (CARL), first proposed by Bonifacio and co-workers [2] and recently extended to describe the quantum mechanical motion of atoms in a BEC [3–5]. In CARL the scattered radiation mode and the atomic momentum side modes become macroscopically populated via a collective instability. The weak probe field initiated by noise combines with the pump field to form a weak standing wave which acts as a periodic potential. The centre-of-mass motion of the atoms in this potential results in a density modulation (bunching). This bunching process is then seen by the pump laser as a polarization grating in the atomic medium which results in a stimulated backscattering into the probe field. The resulting increase in the probe intensity further increases the strength of the standing wave potential, resulting in more bunching and backscattering. This mechanism gives rise to an exponential growth of both the probe intensity and the atomic bunching.

Till now, experiments in a good-cavity regime have been performed only at high temperatures of the atomic sample [6]. However, experiments at MIT [7] and at LENS [8] with BECs have demonstrated that the CARL instability can also play an important role in the case in which laser light is scattered into the vacuum modes of the electromagnetic field in the absence of the cavity. The experiments have demonstrated the formation of atomic matter waves in a cigar-shaped BEC, together with highly directional scattering of light along the major axis of the condensate. This emission has been interpreted in reference [7] as superradiant Rayleigh scattering.

From an experimental point of view the possibility of having BECs inside an optical cavity would allow a verification of the quantum regime of CARL and an exploration of his entanglement properties. As a matter of fact the realization of the quantum CARL in the good-cavity regime offers the possibility of parametrically amplifying atomic and optical waves, as well as of optically manipulating matter-wave coherence properties and generating entanglement between atomic and optical fields.

In the experiments on CARL, the condensate is lightened by the laser after the trap has been switched off, so that the atoms can be assumed unbound by any magnetic or optical confining potential. Furthermore, the atom–atom interaction may be neglected since the experiments are usually performed during the expansion of the condensate. In the limit of undepleted atomic ground state (linear regime) and unsaturated probe field, the quantum CARL Hamiltonian reduces to that of three coupled bosonic modes [5, 9]. The first two modes correspond to atoms having lost or gained a quantum recoil momentum $2\hbar k$ in the two-photon Bragg scattering between the pump and the probe, whereas the third mode corresponds to the photons of the probe field. The quantum statistical properties of the side modes of the condensate can be manipulated and, in particular, a strong quantum mechanical entanglement can develop between the optical and matter-wave fields, as well as between matter-wave side modes [5, 10]. In this paper we analyse in detail the resulting three-mode dynamics, showing the appearance of three-mode and two-mode entanglement, either for an ideal cavity as well as taking into account the losses. Our results have an all-optical analogue in the field of nonlinear quantum optics, where the appearance of three-mode entanglement has been predicted for systems of five optical modes interacting parametrically in a nonlinear crystal [11].

The paper is structured as follows. In section 2 the Hamiltonian model for the system under consideration is briefly reviewed, and the solution of the Heisenberg equations in the linear regime is presented. In section 3 the dynamics of the system is analysed for an ideal cavity, and the entanglement properties of the evolved state are studied. The appearance of a fully inseparable three-mode entangled state is demonstrated, as well as the generation of maximally entangled atom–atom and atom–field two-mode states. In section 4 the effects of losses in the optical cavity are considered solving the master equation in terms of the Wigner function of the three modes. Section 5 closes the paper with some concluding remarks.

2. The Hamiltonian model

We consider a 1D (one-dimensional) geometry in which the off-resonant laser pulse is directed along the symmetry z-axis of an elongated BEC (see figure 1). We define all adimensional variables. The scattered and the incident wave vectors are $\vec{k}_s \approx -\vec{k}$. The dimensionless position and velocity of the jth atom along the axis $\hat{z}$ directed along $\vec{k}$ are $\theta_j = 2\vec{k} \cdot \vec{z}_j = 2kz_j$ and $p_j = mv_{zj}/2\hbar k$. The interaction time in units of the collective recoil bandwidth, $\rho\omega_r$, is

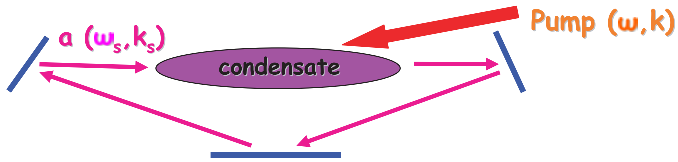

Figure 1. Schematic of the system under investigation.

$\tau = \rho\omega_r t$, where $\omega_r = 2\hbar k^2/M$ is the recoil frequency, M is the atomic mass and

$$\rho = \left(\frac{\Omega_0}{2\Delta_{20}}\right)^{2/3}\left(\frac{\omega\mu^2 n_s}{\hbar\epsilon_0\omega_r^2}\right)^{1/3} \tag{1}$$

is the collective CARL parameter. $\Omega_0 = \mu\mathcal{E}_0/\hbar$ is the Rabi frequency of the pump, $n_s = N/V$ is the average atomic density of the sample (containing N atoms in a volume V), μ is the dipole matrix element and ϵ_0 is the permittivity of the free space. In the second quantized model for CARL [4, 5, 9] the atomic field operator $\hat{\Psi}(\theta)$ obeys the bosonic equal-time commutation relations $[\hat{\Psi}(\theta), \hat{\Psi}^\dagger(\theta')] = \delta(\theta-\theta')$, $[\hat{\Psi}(\theta), \hat{\Psi}(\theta')] = [\hat{\Psi}^\dagger(\theta), \hat{\Psi}^\dagger(\theta')] = 0$ and the normalization condition $\int_0^{2\pi} \mathrm{d}\theta\,\hat{\Psi}(\theta)^\dagger\hat{\Psi}(\theta) = \hat{N}$. We assume that the atoms are delocalized inside the condensate and that, at zero temperature, the momentum uncertainty $\sigma_{p_z} \approx \hbar/\sigma_z$ can be neglected with respect to $2\hbar k$. When $\sigma_z \approx L$, where L is the size of the condensate, this approximation is valid for $L \gg \lambda$, where λ is the wavelength of the incident radiation. So we can introduce creation and annihilation operators for the atoms of a definite momentum p, i.e. $\hat{\Psi}(\theta) = \sum_m c_m\langle\theta|m\rangle$, where $p|m\rangle = m|m\rangle$ (with $m = -\infty, \ldots, \infty$), $\langle\theta|m\rangle = (2\pi)^{-1/2}\exp(\mathrm{i}m\theta)$ and c_m are bosonic operators obeying the commutation relations $\left[c_m, c_{m'}^\dagger\right] = \delta_{mm'}$ and $[c_m, c_{m'}] = 0$. The Hamiltonian in this case is [3, 12]

$$\hat{H} = \sum_{n=-\infty}^{\infty}\left\{\frac{n^2}{\rho}c_n^\dagger c_n + \mathrm{i}\sqrt{\rho/2N}\left(a^\dagger c_n^\dagger c_{n+1} - \mathrm{h.c.}\right)\right\} - \Delta a^\dagger a \tag{2}$$

where $\Delta = (\omega - \omega_s)/\rho\omega_r$ is the pump-probe detuning and $\omega_s = ck_s$ is the frequency of the cavity mode. The Heisenberg equations for c_n and a are

$$\frac{\mathrm{d}c_n}{\mathrm{d}\tau} = -\mathrm{i}[c_n, \hat{H}] = -\mathrm{i}\frac{n^2}{\rho}c_n + \sqrt{\rho/2N}(a^\dagger c_{n+1} - ac_{n-1}) \tag{3}$$

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\mathrm{i}[a, \hat{H}] = \mathrm{i}\Delta a + \sqrt{\rho/2N}\sum_{n=-\infty}^{\infty} c_n^\dagger c_{n+1}. \tag{4}$$

The source of the field equation (4) is the bunching operator, defined by $\hat{B} = (1/N)\int_0^{2\pi}\mathrm{d}\theta\,\hat{\Psi}(\theta)^\dagger\,\mathrm{e}^{-\mathrm{i}\theta}\,\hat{\Psi}(\theta) = (1/N)\sum_n c_n^\dagger c_{n+1}$. We note that equations (3) and (4) conserve the number of atoms, i.e. $\sum_n c_n^\dagger c_n = \hat{N}$, and the total momentum, $\hat{Q} = a^\dagger a + \sum_n nc_n^\dagger c_n$.

Let us now consider the equilibrium state with no probe field and all the atoms in the same initial momentum state n_0, i.e. $c_n \approx \sqrt{N}\,\mathrm{e}^{-\mathrm{i}n^2\tau/\rho}\delta_{n,n_0}$. This is equivalent to assuming the temperature of the system equal to zero and all the atoms moving with the same momentum $n_0(2\hbar\vec{k})$, without spread. This approximation neglects both the depletion that occurs as atoms are transferred into the side modes and the cross-phase modulation between the condensate and the scattered field. This is the matter-wave-optics analogue of the familiar classical undepleted pump approximation of nonlinear optics. Hence we treat all strongly populated modes classically and all weakly populated modes quantum mechanically. The system is unstable for certain values of the detuning Δ. In fact, by linearizing equations (3) and (4) around the equilibrium state, the only equations depending linearly on the radiation field are those for c_{n_0-1} and c_{n_0+1}. Hence, in the linear regime, the only transitions involved are those from the state n_0 towards the levels $n_0 - 1$ and $n_0 + 1$. Introducing the operators

$$a_1 = c_{n_0-1} \exp\left(\mathrm{i}\left(n_0^2\tau/\rho + \Delta\tau\right)\right) \qquad a_2 = c_{n_0+1} \exp\left(\mathrm{i}\left(n_0^2\tau/\rho - \Delta\tau\right)\right)$$
$$a_3 = a\exp(-\mathrm{i}\Delta\tau), \tag{5}$$

the atomic field operator reduces to

$$\hat{\Psi}(\theta) \approx \frac{1}{\sqrt{2\pi}}\{\sqrt{N} + a_1(\tau)\exp(-\mathrm{i}(\theta+\Delta\tau)) + a_2(\tau)\exp(\mathrm{i}(\theta+\Delta\tau))\}\exp\left(\mathrm{i}\left(n_0\theta - n_0^2\tau/\rho\right)\right) \tag{6}$$

and equations (3) and (4) reduce to the linear equations for three coupled harmonic oscillator operators [10],

$$\frac{\mathrm{d}a_1^\dagger}{\mathrm{d}\tau} = -\mathrm{i}\delta_- a_1^\dagger + \sqrt{\rho/2}a_3 \qquad \frac{\mathrm{d}a_2}{\mathrm{d}\tau} = -\mathrm{i}\delta_+ a_2 - \sqrt{\rho/2}a_3 \qquad \frac{\mathrm{d}a_3}{\mathrm{d}\tau} = \sqrt{\rho/2}\left(a_1^\dagger + a_2\right), \tag{7}$$

with Hamiltonian

$$\hat{H} = \delta_+ a_2^\dagger a_2 - \delta_- a_1^\dagger a_1 + \mathrm{i}\sqrt{\rho/2}\left[\left(a_1^\dagger + a_2\right)a_3^\dagger - \left(a_1 + a_2^\dagger\right)a_3\right], \tag{8}$$

where $\delta_\pm = \delta \pm 1/\rho$ and $\delta = \Delta + 2n_0/\rho = (\omega - \omega_s + 2n_0\omega_r)/\rho\omega_r$. Hence, the dynamics of the system is that of three parametrically coupled harmonic oscillators a_1, a_2 and a_3 [13], which obey the commutation rules $[a_i, a_j] = 0$ and $\left[a_i, a_j^\dagger\right] = \delta_{ij}$ for $i, j = 1, 2, 3$. Note that the Hamiltonian (8) commutes with the constant of motion

$$\hat{Q} = a_2^\dagger a_2 - a_1^\dagger a_1 + a_3^\dagger a_3. \tag{9}$$

The exact solution of equations (7) can be obtained using Laplace transform [5, 13]. After some algebra we have

$$a_1^\dagger = \mathrm{e}^{-\mathrm{i}\delta\tau}\left[g_1 a_1^\dagger(0) + g_2 a_2(0) + g_3 a_3(0)\right] \tag{10}$$

$$a_2 = \mathrm{e}^{-\mathrm{i}\delta\tau}\left[h_1 a_1^\dagger(0) + h_2 a_2(0) + h_3 a_3(0)\right] \tag{11}$$

$$a_3 = \mathrm{e}^{-\mathrm{i}\delta\tau}\left[f_1 a_1^\dagger(0) + f_2 a_2(0) + f_3 a_3(0)\right] \tag{12}$$

where the explicit expressions of f_i, g_i and h_i are given in the appendix, while the initial values verify the initial conditions for a_i. The functions f_i, g_i and h_i are the sum of three terms proportional to $\mathrm{e}^{\mathrm{i}\lambda_j\tau}$, where λ_j are the complex roots of the cubic equations:

$$(\lambda - \delta)(\lambda^2 - 1/\rho^2) + 1 = 0. \tag{13}$$

The characteristic equation (13) has either three real solutions, or one real and a pair of complex conjugate solutions. In the first case, the system is stable and exhibits only small oscillations around its initial state. In the second case, the system is unstable and grows exponentially, even from noise.

3. Three-mode entanglement

The evolution operator $U(\tau) = \exp(-\mathrm{i}H\tau)$, where H is given by equation (8), can be disentangled into those of individual operators [5]. This allows us to calculate how the state $|\psi_\tau\rangle$ evolves from the vacuum state $|0, 0, 0\rangle$. The calculation yields

$$|\psi_\tau\rangle = \frac{1}{\sqrt{1 + \langle n_1\rangle}}\sum_{n,m=0}^{\infty}\alpha_1^m\alpha_2^n\sqrt{\frac{(m+n)!}{m!n!}}|m+n, n, m\rangle, \tag{14}$$

where

$$\alpha_1 = \frac{f_1 g_1^*}{1+\langle n_1\rangle} \qquad \alpha_2 = \frac{h_1 g_1^*}{1+\langle n_1\rangle} \qquad |\alpha_{1,2}|^2 = \langle n_{3,2}\rangle/(1+\langle n_1\rangle). \tag{15}$$

The state in equation (14) is a fully inseparable three-mode Gaussian state. This property can be easily demonstrated by evaluating the characteristic function

$$\chi_{id}(\xi_1,\xi_2,\xi_3) = \mathrm{Tr}[|\psi_\tau\rangle\langle\psi_\tau|D_1(\xi_1)D_2(\xi_2)D_3(\xi_3)] = \exp\left[-\frac{1}{2}\sum_{j=1}^{3}|\Gamma_j|^2\right], \tag{16}$$

where ξ_j are complex numbers, $D_j(\xi_j) = \exp\left(\xi_j a_j^\dagger - \xi_j^* a_j\right)$ is a displacement operator for the jth mode, and the Γ_j are given by

$$\Gamma_1 = g_1\xi_1 - h_1\xi_2^* - f_1\xi_3^* \qquad \Gamma_2 = -g_2^*\xi_1^* + h_2^*\xi_2 + f_2^*\xi_3 \qquad \Gamma_3 = -g_3^*\xi_1^* + h_3^*\xi_2 + f_3^*\xi_3. \tag{17}$$

Following [14], the characteristic function can be rewritten as

$$\chi_{id}(\xi_1,\xi_2,\xi_3) = \exp\left[-\tfrac{1}{4}\mathbf{x}^T\mathcal{C}\mathbf{x}\right], \tag{18}$$

where $\mathbf{x}^T = (x_1, x_2, x_3, p_1, p_2, p_3)$, $(\cdots)^T$ denotes transposition, $\xi_j = 2^{-1/2}(p_j - \mathrm{i}x_j)$, $j = 1, 2, 3$, and $\mathcal{C}$ denotes the 6×6 covariance matrix of the Gaussian state, whose explicit expression can be easily reconstructed from equation (17). The covariance matrix determines the entanglement properties of $|\psi_\tau\rangle$, in fact, since $|\psi_\tau\rangle$ is Gaussian the positivity of the partial transpose is a necessary and sufficient condition for separability [14], which, in turn, is determined by the positivity of the matrices $\Lambda_j\mathcal{C}\Lambda_\mathbf{j} - \mathrm{i}\mathbf{J}$ where $\Lambda_1 = \mathrm{Diag}(1,1,1,-1,1,1)$, $\Lambda_2 = \mathrm{Diag}(1,1,1,1,-1,1)$, $\Lambda_3 = \mathrm{Diag}(1,1,1,1,1,-1)$ and $\mathbf{J}$ is the symplectic block matrix

$$\begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix},$$

$\mathbf{I}$ being the 3×3 identity matrix. Characteristic equations of the above three matrices can be analytically solved, showing that a negative eigenvalue always appears. Correspondingly, the state (14) is fully inseparable, i.e. not separable for any grouping of the modes.

Two-mode entangled states between the modes 1 and 2 or the modes 1 and 3 can be obtained for interaction times leading to $\langle n_3\rangle \ll \langle n_1\rangle \approx \langle n_2\rangle$ or $\langle n_2\rangle \ll \langle n_1\rangle \approx \langle n_3\rangle$ respectively. In these cases one has

$$|\psi_{1,2}\rangle = \frac{1}{\sqrt{1+\langle n_1\rangle}}\sum_{n=0}^{\infty}\alpha_2^n|n,n,0\rangle \tag{19}$$

$$|\psi_{1,3}\rangle = \frac{1}{\sqrt{1+\langle n_1\rangle}}\sum_{n=0}^{\infty}\alpha_1^n|n,0,n\rangle. \tag{20}$$

The pure states (19) and (20) are maximally entangled bipartite states, as it can be shown by evaluating the reduced density operators $\rho_i = \mathrm{Tr}_1[\rho_{1i}]$, where $\rho_{1i} = |\psi_{1i}\rangle\langle\psi_{1i}|$ and $i = 2, 3$. In fact, in both cases we obtain a thermal state

$$\rho_i = \frac{1}{1+\langle n_i\rangle}\sum_m\left(\frac{\langle n_i\rangle}{1+\langle n_i\rangle}\right)^m|m\rangle\langle m|, \tag{21}$$

for which the von Neumann entropy $S_i = \mathrm{Tr}[\rho_i \ln\rho_i]$ is maximum [15]. In general, the presence of the third mode reduces the entanglement between the other two modes [16]. We also observe that no two-mode entanglement is possible between the states 2 and 3. In practice, there exist two different regimes of CARL dynamics in which the initial vacuum state evolves into a two-mode entangled state [5]. In particular, atom–atom entanglement can be obtained in the limit $\rho \gg 1$ and in a detuned, not fully exponential regime. In contrast, in the limit $\rho < 1$, atom–photon entanglement can be obtained when the average occupation number $\langle n_2 \rangle$ remains smaller than 1. Recently, the atom–field entanglement of the state (20) has been exploited to suggest an interspecies teleportation protocol between a radiation beam and a condensate side beam [17].

4. Dissipative master equation

We have considered so far an ideal optical cavity with no losses. In order to have a more realistic description of the entanglement generation we now take into account losses from the cavity. The dynamics of the system is described by the master equation

$$\dot{\rho} = -\mathrm{i}[\hat{H}, \rho] + 2\kappa L[a_3]\rho, \tag{22}$$

where 2κ is the damping rate and $L[a_3]$ is the Lindblad superoperator $L[a_3]\rho = a_3\rho a_3^\dagger - \frac{1}{2}a_3^\dagger a_3\rho - \frac{1}{2}\rho a_3^\dagger a_3$. The master equation can be transformed into a Fokker–Planck equation for the Wigner function

$$W(\alpha_1, \alpha_2, \alpha_3, \tau) = \int \prod_{i=1}^{3} \frac{\mathrm{d}^2\xi_i}{\pi^2} \exp(\xi_i^*\alpha_i - \alpha_i^*\xi_i)\chi(\xi_1, \xi_2, \xi_3, \tau). \tag{23}$$

Using the differential representation of the Lindblad superoperator the Fokker–Planck equation is given by

$$\frac{\partial W}{\partial \tau} = -\{\mathbf{u}'^T\mathbf{A}\mathbf{u} + \mathbf{u}'^{*T}\mathbf{A}^\dagger\mathbf{u}^*\}W + \mathbf{u}'^T D\mathbf{u}'^* \tag{24}$$

where

$$\mathbf{u}^T = (\alpha_1^*, \alpha_2, \alpha_3) \qquad \mathbf{u}'^T = \left(\frac{\partial}{\partial\alpha_1^*}, \frac{\partial}{\partial\alpha_2}, \frac{\partial}{\partial\alpha_3}\right) \tag{25}$$

$$\mathbf{A} = \begin{pmatrix} \mathrm{i}\delta_- & 0 & -\sqrt{\rho/2} \\ 0 & \mathrm{i}\delta_+ & \sqrt{\rho/2} \\ -\sqrt{\rho/2} & -\sqrt{\rho/2} & \kappa \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix}. \tag{26}$$

The solution of the Fokker–Planck is given by the following convolution,

$$W(\mathbf{u}, \tau) = \int \mathrm{d}^2\mathbf{u}_0\, W(\mathbf{u}_0, 0)G(\mathbf{u}, \tau; \mathbf{u}_0, 0) \tag{27}$$

where $W(\mathbf{u}_0, 0)$ is the Wigner function for the initial state (the vacuum) and the Green function $G(\mathbf{u}, t; \mathbf{u}_0, 0)$ is given by

$$G(\mathbf{u}, \tau; \mathbf{u}_0, 0) = \frac{1}{\pi^3 \det \mathbf{Q}} \exp\{-(\mathbf{u} - \mathrm{e}^{\mathbf{A}\tau}\mathbf{u}_0)^\dagger \mathbf{Q}^{-1}(\mathbf{u} - \mathrm{e}^{\mathbf{A}\tau}\mathbf{u}_0)\} \tag{28}$$

where

$$\mathbf{Q} = \int_0^\tau \mathrm{d}\tau'\, \mathrm{e}^{\mathbf{A}\tau'}\mathbf{D}(\mathrm{e}^{\mathbf{A}\tau'})^\dagger \qquad \mathrm{e}^{\mathbf{A}\tau} = \mathrm{e}^{-\mathrm{i}\delta\tau} \begin{pmatrix} g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \\ f_1 & f_2 & f_3 \end{pmatrix}, \tag{29}$$

and the covariance matrix is given by $\mathbf{C} = \mathbf{Q} + (1/2)\exp(\mathbf{A}\tau)\exp(\mathbf{A}^\dagger\tau)$. In equation (29) the complex functions g_i, h_i and f_i have the same expression reported in the appendix where however δ is replaced by $\delta + \mathrm{i}\kappa$ and the characteristic equation is now

$$(\lambda - \delta - \mathrm{i}\kappa)(\lambda^2 - 1/\rho^2) + 1 = 0. \tag{30}$$

Since the evolution for a lossy cavity maintains the Gaussian character of the Wigner function we may study the entanglement properties at any time. A numerical evaluation of the eigenvalues of $\Lambda_j \mathcal{C} \Lambda_j - \mathrm{i}\mathbf{J}$ shows that they are nonpositive matrices $\forall j$, i.e. the state is fully inseparable at any time and for any value of the cavity loss. Therefore, we conclude that the generation of three-mode entanglement in the linear regime is robust against decoherence induced by losses [18][4].

5. Conclusions

In this paper, we analysed the interaction between a Bose–Einstein condensate and a single-mode quantized radiation field in the presence of a strong far off-resonant pump laser. In the so-called linear regime, i.e. for situations where atomic ground state depletion and saturation of the radiation mode can be neglected, we have demonstrated the generation of atom–atom and atom–field entanglement. We have also taken into account the effects of cavity imperfections and shown that the state remains fully inseparable for any values of cavity loss κ. A systematic study of the effect of losses, including atomic losses, on the entanglement production is in progress and results will be published elsewhere.

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Appendix

The expressions of the quantities f_i, g_i and h_i $(i = 1, 2, 3)$ which appear in the general solution of the linear problem (10)–(12) are given by

$$f_1(\tau) = -\mathrm{i}\sqrt{\frac{\rho}{2}}\sum_{j=1}^{3}(\lambda_j + 1/\rho)\frac{\mathrm{e}^{\mathrm{i}\lambda_j\tau}}{\Delta_j} = g_3(\tau) \tag{A.1}$$

$$f_2(\tau) = -\mathrm{i}\sqrt{\frac{\rho}{2}}\sum_{j=1}^{3}(\lambda_j - 1/\rho)\frac{\mathrm{e}^{\mathrm{i}\lambda_j\tau}}{\Delta_j} = -h_3(\tau) \tag{A.2}$$

$$f_3(\tau) = \sum_{j=1}^{3}\left(\lambda_j^2 - 1/\rho^2\right)\frac{\mathrm{e}^{\mathrm{i}\lambda_j\tau}}{\Delta_j} \tag{A.3}$$

[4] Note that in order to model cavity losses, we are considering a zero temperature purely dissipative environment, which is appropriate at optical frequencies. In these conditions (no thermal fluctuations), the robustness of entanglement is already known for the two-mode case [18].

$$g_1(\tau) = \sum_{j=1}^{3} [(\lambda_j - \delta)(\lambda_j + 1/\rho) - \rho/2] \frac{e^{i\lambda_j \tau}}{\Delta_j} \tag{A.4}$$

$$g_2(\tau) = -\frac{\rho}{2} \sum_{j=1}^{3} \frac{e^{i\lambda_j \tau}}{\Delta_j} = -h_1(\tau) \tag{A.5}$$

$$h_2(\tau) = \sum_{j=1}^{3} [(\lambda_j - \delta)(\lambda_j - 1/\rho) + \rho/2] \frac{e^{i\lambda_j \tau}}{\Delta_j} \tag{A.6}$$

where $\Delta_j = \lambda_j(3\lambda_j - 2\delta) - 1/\rho^2$ and λ_1, λ_2 and λ_3 are the roots of the cubic equation (13).

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