

## A quantum model for collective recoil lasing

R. BONIFACIO<sup>1</sup>, M. M. COLA<sup>1</sup>, N. PIOVELLA<sup>1</sup> and G. R. M. ROBB<sup>2</sup>

<sup>1</sup> *Dipartimento di Fisica, Università degli Studi di Milano, INFN & INFN  
Via Celoria 16, Milano I-20133, Italy*

<sup>2</sup> *Department of Physics, University of Strathclyde - Glasgow, G4 0NG, UK*

received 19 July 2004; accepted in final form 29 October 2004

published online 8 December 2004

PACS. 42.50.Fx – Cooperative phenomena in quantum optical systems.

PACS. 41.60.Cr – Free-electron lasers.

PACS. 42.50.Vk – Mechanical effects of light on atoms, molecules, electrons, and ions.

**Abstract.** – Free Electron Laser (FEL) and Collective Atomic Recoil Laser (CARL) are described by the same model of classical equations for properly defined scaled variables. These equations are extended to the quantum domain describing the particle’s motion by a Schrödinger equation coupled to a self-consistent radiation field. The model depends on a single collective parameter  $\bar{\rho}$  which represents the maximum number of photons emitted per particle. We demonstrate that the classical model is recovered in the limit  $\bar{\rho} \gg 1$ , in which the Wigner function associated to the Schrödinger equation obeys to the classical Vlasov equation. On the contrary, for  $\bar{\rho} \leq 1$ , a new quantum regime is obtained in which both FELs and CARLs behave as a two-state system coupled to the self-consistent radiation field and described by Maxwell-Bloch equations.

*Introduction.* – Apparently very different systems as High-Gain Free Electron Laser (FEL) [1] and Collective Atomic Recoil Laser (CARL) [2] exhibit similar behaviors, showing self-bunching and exponential enhancement of the emitted radiation. Originally conceived in a semiclassical framework, they can be as well described quantum-mechanically [3–8]. However, it is not explicitly evident how to obtain the classical limit starting from the quantum description. First attempts to give a quantum description of the FEL have been proposed in the ’80s, starting from a canonical quantization of the  $N$ -particle Hamiltonian in the Heisenberg picture, to study photon statistics and quantum initialization from vacuum in the linear regime [4, 9]. In 1988, Preparata proposed a quantum field theory of FEL [3], in which he has shown that, for  $N \gg 1$ , the FEL dynamics is solved by a single-electron Schrödinger equation coupled to a self-consistent radiation mode. The same model has been recently obtained to describe CARL from a Bose-Einstein condensate (BEC) at zero temperature [6, 8]. Furthermore, it has been also proved experimentally [10–12] that CARL in a BEC exhibits quantum recoil effects when the average recoil velocity remains less than the photon recoil limit.

In this letter we start from the classical model describing both CARLs and FELs and we extend it to the quantum realm showing the correspondence between the Preparata model [3] and the CARL-BEC model [10]. In particular, it is possible to derive an equation for the Wigner function of the  $N$ -particle system. The Wigner function obeys to a finite-difference equation which reduces to the classical Vlasov equation [13] in the limit in which the maximum number of photons emitted per particle is much larger than unity. In the opposite limit, both CARLs and FELs behave as a two-state system [14–16] described by the well-known Maxwell-Bloch equations [17].

*CARL-FEL model.* – Apparently, the physics of FEL and CARL appears to be quite different. The first describes a relativistic high-current electron beam with energy  $mc^2\gamma_0$ , injected in a magnet (“wiggler”) with a transverse, static magnetic field  $B_w$  and periodicity  $\lambda_w$ , which radiates in the forward direction at the wavelength  $\lambda \sim \lambda_w(1 + a_w^2)/2\gamma_0^2$ , where  $a_w = eB_w/mc^2k_w$  is the wiggler parameter and  $k_w = 2\pi/\lambda_w$ . Instead, CARL consists of a collection of two-level atoms in a high- $Q$  ring cavity [18] driven by a far-detuned laser pump of frequency  $\omega_p$  which radiates at the frequency  $\omega \sim \omega_p$  in the direction opposite to the pump. In both cases the radiation process arises from a collective instability which originates a symmetry breaking in the spatial distribution, *i.e.* a self-bunching of particles which group in regions smaller than the wavelength.

It can be shown that, under suitable conditions and introducing proper dimensionless variables, the dynamics of both FELs and CARLs is described by the following Hamiltonian [5]:

$$H = \sum_{j=1}^N \left[ \frac{p_j^2}{\bar{\rho}} + i\sqrt{\frac{\bar{\rho}}{2N}} (a^\dagger e^{-i\theta_j} - \text{h.c.}) \right] - \frac{\delta}{\bar{\rho}} a^\dagger a, \quad (1)$$

where  $\theta_j$  and  $p_j$  are the phase operator of the  $j$ -th particle and its conjugate momentum operator, obeying  $[\theta_j, p_{j'}] = i\delta_{jj'}$ . In eq. (1),  $a$  and  $a^\dagger$  are annihilation and creation operators for the forward radiation mode photon, with  $[a, a^\dagger] = 1$ . Notice that the dynamics described by eq. (1) depends only on the parameter  $\bar{\rho}$  and on the detuning  $\delta$ , properly defined for the two systems:

- For FELs,  $\bar{\rho} = q\rho_F$  and  $\delta = q(\gamma_0 - \gamma_r)/\gamma_r$ , where  $q = mc\gamma_r/\hbar k$ ,  $\gamma_r = \sqrt{(\lambda_w/2\lambda)(1 + a_w^2)}$  is the resonant energy and  $\rho_F = (1/\gamma_r)(a_w/4ck_w)^{2/3}(e^2n/m\epsilon_0)^{1/3}$  is the BPN parameter for a FEL [1],  $\theta = (k_w + k)z - ckt$ ,  $p = q(\gamma - \gamma_0)/2\gamma_r$  and  $k = 2\pi/\lambda$ .
- For CARLs,  $\bar{\rho} = \rho_C$  and  $\delta = (\omega_p - \omega)/\omega_R$ , where  $\omega_R = 2\hbar k^2/m$  is the recoil frequency,  $\rho_C = (S_0/\omega_R)^{2/3}(\omega d^2 n/2\hbar\epsilon_0)^{1/3}$ ,  $S_0 = \Delta\Omega/[2(\Gamma^2 + \Delta^2 + \Omega^2)]$ ,  $\Omega$  is the pump Rabi frequency,  $\Delta$  is the pump-atom detuning,  $\Gamma$  is the natural decay constant of the atomic transition and  $d$  is the dipole matrix element [19]. Finally,  $\theta = 2kz$  and  $p = mv_z/2\hbar k$ , where  $v_z$  is the longitudinal atomic velocity.

In these definitions,  $n = N/V$  is the particle density in the radiation volume  $V$  and  $m$  is the particle mass. Notice that in both cases  $\bar{\rho}$  scales as  $n^{1/3}$ , *i.e.* as the reciprocal of the inter-particle distance. Introducing  $\bar{p}_j = (2/\bar{\rho})p_j$  and  $A = (2/N\bar{\rho})^{1/2}a$ , the Heisenberg equations associated with eq. (1) are [1, 2]

$$\frac{d\theta_j}{d\tau} = \bar{p}_j, \quad (2)$$

$$\frac{d\bar{p}_j}{d\tau} = -(Ae^{i\theta_j} + \text{c.c.}), \quad (3)$$

$$\frac{dA}{d\tau} = \frac{1}{N} \sum_{i=1}^N e^{-i\theta_i} + \frac{i\delta}{\bar{\rho}} A, \quad (4)$$

where  $j = 1 \dots N$  and  $\tau = \omega_R\rho_C t$  for CARL whereas  $\tau = (4\pi\rho_F/\lambda_w)z$  for FEL. Considering the operators  $\theta_j$ ,  $\bar{p}_j$  and  $A$  in eqs. (2)-(4) as c-numbers, one obtains the well-known classical description for FEL and CARL. The scaling of eqs. (2)-(4) is called “universal” in the sense that, assuming resonance (*i.e.*  $\delta = 0$ ), the equations do not contain any parameter. Hence, the scaling law of the various physical quantities can be obtained from their definition in terms of  $\bar{\rho}$ . In fact, solving eqs. (2)-(4) with an arbitrary small initial field  $A_0$  and  $\delta = 0$ , the dimensionless

intensity  $|A|^2$  reaches a maximum value of the order of unity [1], which corresponds to an average number of photons  $|a|^2$  of the order of  $N\bar{\rho}$ , so that  $\bar{\rho}$  is the maximum photon number per particle, both for FELs and CARLs. Hence, it is expected that when  $\bar{\rho} \gg 1$  the system behaves classically, whereas for  $\bar{\rho} \leq 1$  quantum effects become relevant. Notice that  $|A|^2 + N^{-1} \sum_j \bar{p}_j$  is a constant of motion in eqs. (2)-(4), *i.e.* the radiated intensity is due to the average recoil.

*Quantum CARL-FEL model.* – In ref. [3], Preparata, using quantum field theory, has shown that the collective dynamics of the system of  $N \gg 1$  electrons in an FEL can be described by means of a single complex scalar quantum field whose behavior is governed by a Schrödinger-type equation in the self-consistent radiation field, which originates a pendulum-like potential:

$$i \frac{\partial \psi}{\partial \tau} = -\frac{1}{\bar{\rho}} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{i\bar{\rho}}{2} [Ae^{i\theta} - \text{c.c.}] \psi, \quad (5)$$

$$\frac{dA}{d\tau} = \int_0^{2\pi} d\theta |\psi|^2 e^{-i\theta} + \frac{i\delta}{\bar{\rho}} A, \quad (6)$$

where  $\psi$  is normalized to one, *i.e.*  $\int_0^{2\pi} d\theta |\psi(\theta, \tau)|^2 = 1$ . Note that eq. (5) is the Schrödinger equation associated to the Hamiltonian (1) and eq. (6) corresponds to eq. (4) when the classical average of  $e^{-i\theta}$  is replaced by the quantum ensemble average. Quoting ref. [3], eqs. (5) and (6) are derived if one “formulates the many-electron problem in the language of quantum field theory and uses the large number  $N$  of electrons to evaluate the resulting path integral by saddle-point techniques”. Recently, the same model of eqs. (5) and (6) has been used to describe CARL from a BEC [6, 8, 10]. Hence, we propose the nonlinear system of eqs. (5) and (6) as the quantum extension of the CARL-FEL classical model. We now show that the classical equations (2)-(4) are recovered in the limit  $\bar{\rho} \gg 1$ .

*Wigner function approach.* – Let us consider the standard definition of the Wigner function for a state with wave function  $\psi(\theta, \tau)$ :

$$W(\theta, p, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi e^{i\xi p} \psi^* \left( \theta - \frac{\xi}{2}, \tau \right) \psi \left( \theta + \frac{\xi}{2}, \tau \right), \quad (7)$$

so that

$$\int_{-\infty}^{+\infty} dp W(\theta, p, \tau) = |\psi(\theta, \tau)|^2. \quad (8)$$

One can show that eq. (5) is equivalent to the following finite-difference equation for the quasi-probability distribution  $W(\theta, \bar{p}, \tau)$ :

$$\begin{aligned} & \frac{\partial W(\theta, \bar{p}, \tau)}{\partial \tau} + \bar{p} \frac{\partial W(\theta, \bar{p}, \tau)}{\partial \theta} - \\ & - \frac{\bar{\rho}}{2} [Ae^{i\theta} + \text{c.c.}] \left[ W \left( \theta, \bar{p} + \frac{1}{\bar{\rho}}, \tau \right) - W \left( \theta, \bar{p} - \frac{1}{\bar{\rho}}, \tau \right) \right] = 0. \end{aligned} \quad (9)$$

Using eq. (8), eq. (6) becomes

$$\frac{dA}{d\tau} = \int_{-\infty}^{+\infty} d\bar{p} \int_0^{2\pi} d\theta W(\theta, \bar{p}, \tau) e^{-i\theta} + \frac{i\delta}{\bar{\rho}} A. \quad (10)$$

We underline again that eqs. (9) and (10) are equivalent to eqs. (5) and (6) using the Wigner function representation. However, the Wigner function behaves as a classical distribution in

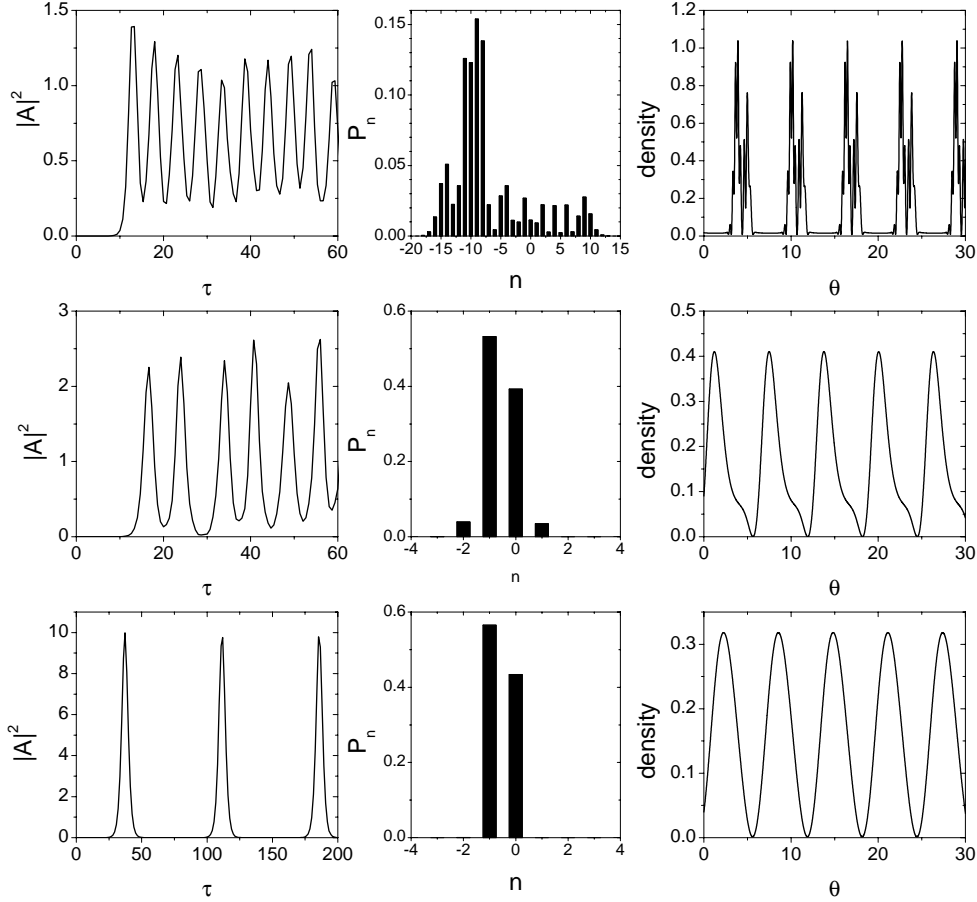


Fig. 1 – Numerical solution of eqs. (13) and (14) for  $\bar{\rho} = 10$  (first row),  $\bar{\rho} = 1$  (second row) and  $\bar{\rho} = 0.2$  (third row). The other parameters are  $\delta = 1$ ,  $A(0) = 10^{-4}$  and  $c_n(0) = \delta_{n0}$ . Left column: dimensionless radiation intensity  $|A|^2$  vs.  $\tau$ ; central and right column: occupation probabilities  $P_n = |c_n|^2$  vs.  $n$  and density distribution  $|\psi|^2$  vs.  $\theta$ , for  $\tau$  near the first maximum of  $|A|^2$ .

the limit  $\bar{\rho} \gg 1$ . In fact, in the right-hand side of eq. (9), the incremental ratio  $[W(\theta, \bar{p} + \epsilon) - W(\theta, \bar{p} - \epsilon)]/(2\epsilon) \rightarrow \partial W(\theta, \bar{p})/\partial \bar{p}$  when  $\epsilon = 1/\bar{\rho} \rightarrow 0$ . Hence, for  $\bar{\rho} \gg 1$ , eq. (9) becomes the Vlasov equation:

$$\frac{\partial W(\theta, \bar{p}, \tau)}{\partial \tau} + \bar{p} \frac{\partial W(\theta, \bar{p}, \tau)}{\partial \theta} - [Ae^{i\theta} + \text{c.c.}] \frac{\partial W(\theta, \bar{p}, \tau)}{\partial \bar{p}} = 0, \quad (11)$$

so that in this limit the Wigner function behaves as a definite positive, classical distribution function. Equations (10) and (11) are equivalent to the classical equations (2)-(4). This means that the particles behave classically, following a Newtonian motion, when  $\bar{\rho} \gg 1$ , *i.e.* when the average number of photons scattered per particle is much larger than unity. In this limit, the quantum recoil effects due to the single-photon scattering process are negligible. On the contrary, a quantum regime of CARL or FEL occurs when  $\bar{\rho} \leq 1$ , in which each particle scatters only one photon. In fact, expanding the wave function in Fourier series as

$$\psi(\theta, \tau) = \frac{1}{\sqrt{2\pi}} \sum_n c_n(\tau) e^{in(\theta + \frac{\delta}{\bar{\rho}}\tau)}, \quad n = -\infty, \dots, +\infty, \quad (12)$$

and inserting this ansatz in eqs. (5) and (6), one can easily obtain the following closed set of equations for  $\varrho_{m,n}(\tau) = c_m^*(\tau)c_n(\tau)$ :

$$\begin{aligned} \frac{d\varrho_{m,n}}{d\tau} = & \frac{i}{\bar{\rho}}(m-n)(\delta+m+n)\varrho_{m,n} + \\ & + \frac{\bar{\rho}}{2} [\bar{A}(\varrho_{m+1,n} - \varrho_{m,n-1}) + \bar{A}^*(\varrho_{m,n+1} - \varrho_{m-1,n})], \end{aligned} \quad (13)$$

$$\frac{d\bar{A}}{d\tau} = \sum_{n=-\infty}^{\infty} \varrho_{n-1,n}, \quad (14)$$

where  $\bar{A} = Ae^{-i(\delta/\bar{\rho})\tau}$ . These equations are equivalent to eqs. (5) and (6) for the density matrix in the momentum representation and have been discussed in ref. [15]. Figure 1 shows the numerical solution of eqs. (13) and (14) for  $\bar{\rho} = 10$  (first row),  $\bar{\rho} = 1$  (second row) and  $\bar{\rho} = 0.2$  (third row). The other parameters are  $A(0) = 10^{-4}$ ,  $c_n(0) = \delta_{n0}$  and  $\delta = 1$ , which corresponds to a single-photon scattering recoil. In the first column  $|A|^2$  is plotted as a function of  $\tau$ . The central column shows the occupation probabilities  $P_n = |c_n|^2$  vs.  $n$ , whereas the right column shows the density distribution  $|\psi|^2$  vs.  $\theta$ , for a value of  $\tau$  near the first maximum of  $|A|^2$ . We note that for  $\bar{\rho} = 10$ , the system behaves classically: the momentum states are occupied in a range of the order of  $\bar{\rho}$  and the average momentum is  $\langle p \rangle \approx -\bar{\rho}$ , as can be seen from the first row of fig. 1. Furthermore, the particle distribution shows periodic narrow peaks of density. When  $\bar{\rho} = 1$ , the mainly occupied momentum states are those for  $n = 0$  and  $n = -1$ , corresponding to particles in the initial state or in the recoil state, respectively. Finally, when  $\bar{\rho} \ll 1$ , the dynamics is that of a pure two-level system. In this limit, if the initially occupied state is the  $n$ -th momentum state, the only two momentum states involved in the interaction are those for  $n$  and  $n - 1$ , so that eqs. (13) and (14), after defining the ‘‘polarization’’  $S_n = \varrho_{n-1,n}$  and the ‘‘population difference’’  $D_n = \varrho_{n,n} - \varrho_{n-1,n-1}$ , reduce to the Maxwell-Bloch equations for a two-state system [17]:

$$\frac{dS_n}{d\tau'} = -i\Delta_n S_n + \frac{1}{2}A'D_n, \quad (15)$$

$$\frac{dD_n}{d\tau'} = -\left(A'S_n^* + A'^*S_n\right), \quad (16)$$

$$\frac{dA'}{d\tau'} = S_n, \quad (17)$$

where  $\Delta_n = (\delta - 1 + 2n)/\bar{\rho}^{3/2}$ ,  $A' = \sqrt{\bar{\rho}}\bar{A}$  and  $\tau' = \sqrt{\bar{\rho}}\tau$ . With this new scaling and assuming resonance (*i.e.*  $\Delta_n = 0$ ), eqs. (15)-(17) do not contain any parameter. Hence, the characteristic timescale is ruled by  $\sqrt{n}$  instead of  $n^{1/3}$  as in the classical case. The quantum regime for CARLs and FELs is analog to the coherent spontaneous-emission regime predicted quantum-mechanically for a two-level system in ref. [20], where a series of optical ‘‘ $2\pi$ -pulses’’ are generated. In fact, assuming resonance (*i.e.*  $\Delta_n = 0$ ),  $A'$  and  $S_n$  are real. Hence, we can introduce the ‘‘Bloch angle’’  $\phi$  such that  $S_n = (1/2)\sin\phi$ ,  $D_n = \cos\phi$ . Then, eqs. (15)-(17) reduce to a pendulum equation  $d^2\phi/d\tau'^2 = (1/2)\sin\phi$  and  $d\phi/d\tau' = A'$ . Hence, in the quantum regime, the dynamics is that of a pendulum moving away from the unstable equilibrium point ( $\phi = 0$ ) and undergoing periodically a complete revolution (‘‘ $2\pi$ -pulse’’) with angular velocity  $A'$ .

Finally, we note that, adopting the same scaling of eqs. (15)-(17) in eq. (5), this can be interpreted as a Schrödinger equation for a single particle with a ‘‘mass’’  $\bar{\rho}^{3/2}$  in a self-consistent pendulum potential. This provides an intuitive interpretation of the classical limit,

that holds when the particle's "mass" is large. The strong differences between the quantum and classical regimes are evident from fig. 1.

*Conclusions.* – In this letter we presented a unified quantum model that stands for apparently very different systems as FEL and CARL. The dynamics is described by a Schrödinger equation in a self-consistent pendulum potential and is ruled by a unique parameter  $\bar{\rho}$  which represents the maximum number of photons scattered per particle and the maximum momentum recoil in units of the photon recoil momentum. The Schrödinger equation can be transformed into an exact equation for the Wigner quasi-probability distribution. The main results are the following: i) The classical model is recovered in the limit  $\bar{\rho} \gg 1$ ; this because the finite-difference equation for the Wigner function reduces to the classical Vlasov equation. ii) In the limit  $\bar{\rho} \leq 1$ , a completely different dynamical regime occurs (see fig. 1): due to momentum quantization the system reduces to only two momentum states obeying to the Maxwell-Bloch equations which describe the dynamics of a two-level atomic system coupled to a coherent field.

\* \* \*

One of us (RB) has to acknowledge a big mistake he made almost 20 years ago when G. PREPARATA, a well-known field theorist, presented to him his general "Quantum Field Theory of a Free Electron Laser" [3]. RB did not understand this work thinking it was incorrect. On the contrary, Preparata's theory was perfectly correct, as recognized in this letter, which is dedicated to his memory.

## REFERENCES

- [1] BONIFACIO R., PELLEGRINI C. and NARDUCCI L., *Opt. Commun.*, **50** (1984) 373.
- [2] BONIFACIO R. and DE SALVO SOUZA L., *Nucl. Instrum. Methods Phys. Res. A*, **341** (1994) 360; BONIFACIO R., DE SALVO SOUZA L., NARDUCCI L. and D'ANGELO E. J., *Phys. Rev. A*, **50** (1994) 1716.
- [3] PREPARATA G., *Phys. Rev. A*, **38** (1988) 233.
- [4] BONIFACIO R. and CASAGRANDE F., *Opt. Commun.*, **50** (1984) 251.
- [5] BONIFACIO R., *Opt. Commun.*, **146** (1998) 236.
- [6] MOORE M. G., ZOBAY O. and MEYSTRE P., *Phys. Rev. A*, **60** (1999) 1491.
- [7] LING H. Y., PU H., BAKSMATY L. and BIGELOW N. P., *Phys. Rev. A*, **63** (2001) 053810.
- [8] PIOVELLA N., COLA M. and BONIFACIO R., *Phys. Rev. A*, **67** (2003) 013817.
- [9] BECKER W. and MCIVER J. K., *Phys. Rev. A*, **28** (1983) 1838.
- [10] BONIFACIO R., CATALIOTTI F. S., COLA M., FALANI L., FORT C., PIOVELLA N. and INGUSCIO M., *Opt. Commun.*, **233** (2004) 155.
- [11] INOUE S. *et al.*, *Science*, **285** (1999) 571.
- [12] KOZUMA M. *et al.*, *Science*, **286** (1999) 2309; INOUE S. *et al.*, *Nature*, **402** (1999) 641.
- [13] BONIFACIO R. and VERKERK P., *Opt. Commun.*, **124** (1996) 469.
- [14] INOUE S. *et al.*, *Phys. Rev. Lett.*, **85** (2000) 4225.
- [15] PIOVELLA N., GATELLI M. and BONIFACIO R., *Opt. Commun.*, **194** (2001) 167.
- [16] AVETISSIAN H. K. and MKRTCHIAN G. F., *Phys. Rev. E*, **65** (2002) 046505.
- [17] ARECCHI F. T. and BONIFACIO R., *IEEE Quantum Electron.*, **1** (1965) 169.
- [18] The High- $Q$  ring cavity does not appear explicitly in the model, but it is essential in order to assume a lossless interaction of the atoms with a single-mode unidirectional field.
- [19] BONIFACIO R. and DE SALVO SOUZA L., *Opt. Commun.*, **115** (1995) 505.
- [20] BONIFACIO R. and PREPARATA G., *Phys. Rev. A*, **2** (1970) 336.